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CONIC SECTIONS
TREATED GEOMETRICALLY

BY

S. H. HASLAM, B.A.

AND

J. EDWARDS, B.A.

CONIC SECTIONS

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BY

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PREFACE.

THE OBJECT of the following is to place before the student a concise and uniform method of treating Geometrical Conics.

Conics are here defined as *plane loci,* and the general properties are deduced immediately from this definition, by the *aid* of a circle, which we have called the *Auxiliary Circle of a Point,* since its definition includes the circle on the transverse axis of an Ellipse or Hyperbola.

The advantage of this construction will appear especially from the very general theorem of Art. 18; it also shows very clearly the intrinsic relation which the three species of conics bear to one another *and to the circle.* In fact, it leads to a method of *plane projection* which appears to us to be very simple and more powerful than Conical Projection. For want of a better name we have called it Focal Projection.

An introduction has been given to Curvature, Sections of the Cone, Anharmonic Ratio, and Orthogonal Projection; and numerous examples have been appended, which, it is hoped, will give the student a practical acquaintance with the subject.

S. HOLKER HASLAM.

ROCHESTER: *May* 1, 1881.

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CONIC SECTIONS

TREATED GEOMETRICALLY.

CHAPTER I.

GENERAL PROPERTIES OF CONICS.

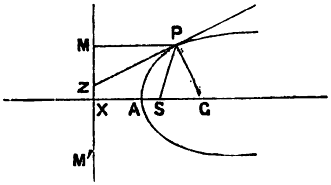
DEFINITIONS.

1. The curve traced out by a point [P], which moves in one plane in such a manner that its distance from a fixed point [S] always bears the same ratio to its perpendicular distance from a fixed straight line [MM′], is called a ‘Conic.’

2. The fixed point [S] is called a ‘Focus,’ and the fixed line [MM′] is called a ‘Directrix.’

3. The fixed ratio is called the ‘Eccentricity’ of the conic.

FIG. 1.



4. The straight line SX through the focus, and perpendicular to the directrix, is called the ‘Axis’ of the conic.

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5. If SX be divided at A, so that SA is to AX in the fixed ratio, then A is a point on the curve, and this point A in which the curve cuts the axis is called a 'Vertex.'

6. A straight line PP′ joining any two points P and P′ on a curve is called a 'Chord.'

7. If PP′ be a chord of a curve, then, in the limiting position which it assumes when the point P′ is indefinitely near to P, it is called a 'Tangent' to the curve at the point P.

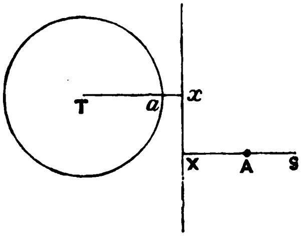
8. A line PG drawn through a point P on a curve perpendicular to the tangent at that point is called a 'Normal' to the curve at that point.

9. Any line TPP′ drawn through a point T, and cutting a curve in P and P′, is called the 'Secant TPP′,' and TP, TP′ are called the 'Segments of the chord through T.'

10. If a pair of tangents be drawn to a curve, the chord, joining their points of contact with the curve, is called the 'Chord of Contact.'

11. If T be any point whatever, and Tx the perpendicular upon the directrix of a conic, and a point a be taken on Tx such that the ratio of Ta to Tx represents the eccentricity, then the circle described with centre T and radius Ta will be called the 'Auxiliary Circle of the point T.'

FIG. 2.



Since the vertex A is a point on the curve, and AX the perpendicular upon the directrix, the ratio of SA to AX must also represent the eccentricity.

There are three species of conics whose forms and properties will be separately considered.

1. When the eccentricity is equal to unity, the form is called a 'Parabola.'

   In this case $SA = AX$, and $Ta = Tx$; therefore the Auxiliary Circle of any point touches the directrix.

2. When the eccentricity is less than unity, the form is called an 'Ellipse.'

   In this case SA is less than AX, and Ta less than Tx; therefore the Auxiliary Circle of any point never meets the directrix.

3. When the eccentricity is greater than unity, the form is called an 'Hyperbola.'

   In this case SA is greater than AX, and Ta greater than Tx; therefore the Auxiliary Circle of every point cuts the directrix.

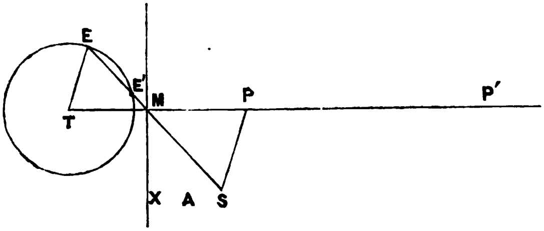
Before tracing the forms of these curves, we shall consider a few properties which are common to all conics.

PROP. 1.—*PROBLEM.*

ART. **1.** *The focus and directrix of a conic being given, to find the points in which any straight line* parallel *to the axis cuts the curve.*

Let the straight line cut the directrix in the point M, and let T be any other point on that straight line. Describe the

FIG. 3.



auxiliary circle of the point T. Join SM, and let it cut the circle in the point E; join TE, and draw SP parallel to TE to meet TM in the point P: then shall P be a point in which the line meets the curve.

From similar triangles, SPM and ETM, we get that

$$SP : PM :: ET : TM$$

But the ratio of the radius ET to the perpendicular TM is equal to the eccentricity; therefore P is a point on the conic.

Q.E.F.

2. In the same way if the straight line SE meet the circle in a second point E′, and SP′ be drawn parallel to TE′, we can find the other point P′ in which TM cuts the conic.

3. If the line SM does not cut the circle, then the line TM does not cut the conic.

EXAMPLES.

1. Show that a parabola only cuts the axis in one point.

2. Show that a straight line parallel to the axis of a parabola only cuts the curve in one point.

3. Show that the auxiliary circle of a vertex passes through a focus.

4. Show that two points on a conic which are equidistant from the directrix are also equidistant from the axis.

5. Show that an ellipse cuts the axis in two points on the same side of the directrix.

6. Show that an hyperbola cuts the axis in two points on opposite sides of the directrix.

7. Show that a straight line parallel to the axis of an ellipse cuts the curve in two points on the same side of the directrix.

8. Show that a straight line parallel to the axis of an hyperbola cuts the curve in two points on opposite sides of the directrix.

9. Show that the centres of all circles which pass through a fixed point and touch a fixed line lie on a parabola.

10. In figure Prop. I. show that the line TMP will always cut the curve in the case of a parabola or hyperbola, however

far the point M is taken from the axis; but that it will not do so in the case of the ellipse.

11. By drawing, in figure Prop. I., the auxiliary circles of the point T corresponding to an ellipse, parabola, and hyperbola, show that the point in which the line TM cuts the parabola lies between the points of section with the ellipse and hyperbola.

12. By means of the last example show that the distance of a point inside a parabola from the focus is less than its distance from the directrix, and greater for a point outside the parabola.

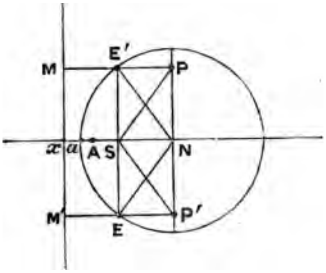
4. The student who has worked out the above examples will already have a tolerably good idea of the forms of the three species of conics and their relation to one another; but each will be discussed separately in the following chapters.

PROP. 2.—PROBLEM.

5. *The focus and directrix of a conic being given, to find the points in which any straight line* perpendicular *to the axis cuts the curve.*

Let the straight line cut the axis in the point N. Describe the auxiliary circle of the point N. Through S draw a line parallel to the directrix, cutting this circle in E and E′. Join NE and NE′, and draw SP and SP′ parallel to NE and NE′,

FIG. 4.



cutting the given line in P and P′: then P and P′ shall be the points required.

For SP is equal to EN, and NX is equal to PM, the perpendicular from P upon the directrix. Hence

$$SP : PM :: EN : NX$$

$$\text{and similarly } SP' : P'M' :: E'N : NX$$

and the ratios EN : NX and E′N : NX = the eccentricity.

Therefore P and P′ are the points in which the given line cuts the curve. Q.E.F.

Cor. We notice in this case that PN = SE, and P′N = SE′.

EXAMPLES.

1. If a straight line perpendicular to the axis meet a conic in the points P and P′, then the line PP′ is bisected by the axis.

2. If a straight line through the focus of a parabola perpendicular to the axis cut the curve in E and E′, then

$$EE' = 4\ AS.$$

3. The tangent at the vertex of a conic is perpendicular to the axis.

4. Show that in the case of the parabola or hyperbola the focus S will always lie within the auxiliary circle of a point N on the axis which lies to the right of the vertex A, however far the point N may be from the directrix.

5. Show that the centres of all circles which touch a given circle and also a given straight line lie on a parabola.

6. If a series of arcs of circles be described passing through two fixed points, and each arc be trisected, all the points of trisection lie on two hyperbolas.

6. *Def.*—A line PN, drawn from any point on a conic perpendicular to the axis and terminated by it, is called an 'Ordinate' to the axis.

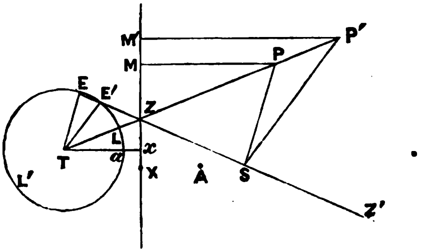
The double ordinate of a conic which passes through the focus is called the 'Latus Rectum.'

PROP. 3.—PROBLEM.

7. *The focus and directrix of a conic being given, to find the points in which it is intersected by* any *straight line.*

Let the given line meet the directrix in the point Z; let T be any other point on that straight line. Describe the auxiliary circle of the point T. Let Tx be the perpendicular upon the directrix. Join SZ, and let it cut the circle in the points E and E′; join TE and TE′, and draw SP and SP′ parallel to TE and TE′ to meet TZ in the points P and P′: then shall P and P′ be the required points.

Fig. 5.



Then because the triangles SPZ, ZPM, are similar to the triangles ETZ, xTZ, each to each,

$$\text{SP} : \text{PZ} :: \text{ET} : \text{TZ}$$
$$\text{and } \text{PZ} : \text{PM} :: \text{TZ} : \text{T}x$$
$$\therefore \text{SP} : \text{PM} :: \text{ET} : \text{T}x$$

Hence P lies on the curve, and in the same way P′ lies on the curve. Q.E.F.

8. Hence we see that any straight line cuts a conic in two points; they are therefore called curves of the second degree.

9. We also notice that corresponding to any secant SEE′ to the auxiliary circle of a point T we have a secant TPP′ to the conic. The points E and E′ may be said to correspond to the points P and P′.

10. *Cor.* 1.—If ZS be produced to Z′, the angles PSZ and P′SZ′ are respectively equal, to the angles TEE′ and TE′E. But the angles TEE′ and TE′E are equal, hence the angles PSZ and P′SZ′ are equal; or, in other words, PS and P′S are equally inclined to SZ.

11. *Cor.* 2.—If the point Z be moved along the directrix until E and E′ coincide and SEE′ becomes a tangent to the circle, the points P and P′ will also coincide, and TPP′ will become a tangent to the conic. The angles TEZ and PSZ will become right angles.

12. The last two theorems being very important, we shall illustrate them by a few examples, which the student is recommended to work out before proceeding further.

EXAMPLES.

1. If PSQ be a chord of a conic passing through the focus, the tangents at P and Q intersect on the directrix, because they both [Cor. 2] intersect the directrix in the point where the line through S perpendicular to PQ meets the directrix.

2. If PSQ, P′SQ′ be two focal chords, then PP′ and QQ′ intersect on the directrix; and also PQ′ and P′Q intersect on the directrix; because they meet the directrix in the same points [Cor. 1] as the bisectors of the angles PSQ′ and Q′SQ.

13. In the course of the proof of [Cor. 2] we have shown that, corresponding to a tangent from S to the auxiliary circle of a point T, there is a tangent from T to the conic which intersects the directrix in the same point Z.

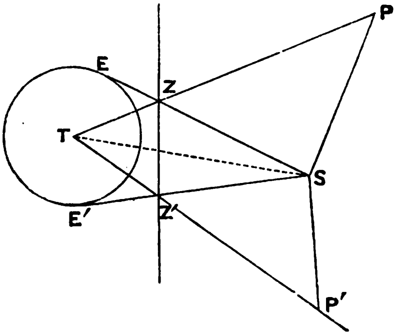
14. Hence to draw a pair of tangents to a conic from a point T we have only to draw a pair of tangents SE, SE′ to the auxiliary circle meeting the directrix in the points Z and Z′. Then TZ and TZ′ will be the tangents to the conic.

PROP. 4.—THEOREM.

15. *If the tangents to a conic from a point* T *touch the curve in* P *and* P′, *then shall* SP *and* SP′ *be equally inclined to* ST.

Let SE, SE′ be the tangents to the auxiliary circle of the point T corresponding to the tangents TP and TP′ and meeting the directrix in the points Z and Z′, then the angles EST and E′ST are equal. But the angles ZSP and Z′SP′ are each right angles [11]. Therefore the whole angle TSP is equal to the whole angle TSP′. Q.E.D.

FIG. 6.



1. If a line be drawn through S parallel to the directrix to meet the tangents TP and TP′ in the points D and D′, prove that

$$SD : SZ :: SD' : SZ'$$

[N.B. ST bisects ZSZ′].

2. If PM be drawn perpendicular to the directrix, prove that the triangle SDZ in the last example is similar to the triangle SPM.

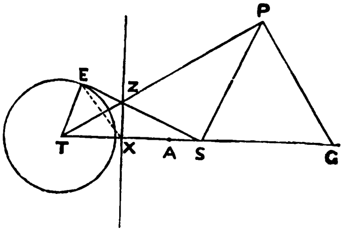
PROP. 5.—*THEOREM.*

16. *If the normal to a conic at a point* P *meet the axis in the point* G, *then*

$$SG : SP :: SA : AX$$

Let the tangent at P meet the axis in T, and let SE be the corresponding tangent to the auxiliary circle of T; join ET and EX.

Fig. 7.



Because the angles TEZ and TXZ are right angles, a circle will go round the quadrilateral TXZE.

∴ the angle ZXE is equal to the angle ZTE.

Hence the angle ZXE is equal to the angle ZPS.

∴ also the complement EXT is equal to the complement SPG.

Again, the angle ETX is equal to the angle PSG.

Hence ETX and GSP are similar triangles.

$$\therefore\ SG : SP :: ET : TX$$
$$:: SA : AX \qquad \text{Q.E.D.}$$

EXAMPLES.

1. Draw the auxiliary circle for the case in which the tangent passes through the point X.

2. If from any point T in the tangent TP to a conic TN be drawn perpendicular to the focal radius SP, and TX perpendicular to the directrix, prove that

SN : TX :: SA : AX (Adams' *Prop.*)

3. If the normal at the point P of a parabola meet the axis in the point G, then SG is equal to SP. In the ellipse SG is less than SP, and in the hyperbola SG is greater than SP.

4. If PG be the normal to a parabola at a point P meeting the axis in the point G, the perpendicular from S upon PG bisects that line.

5. If the normal at a point P of a parabola meet the axis in the point G, and PM be the perpendicular to the directrix, then GPMS is a parallelogram, GM bisects SP, and SP bisects GM.

6. If SL be the ordinate of a parabola which passes through the focus, the tangent and normal at the point L are equally inclined to the axis.

7. If the tangent and normal at a point P of a parabola meet the axis in the points T and G, then

$$SG=SP=ST$$

8. If the tangent and normal at a point P of a parabola meet the axis in the points T and G, and PM be the perpendicular to the directrix, then SPMT is a rhombus and is equal to the parallelogram GPMS; also SM and TP bisect one another at right angles.

DEFINITIONS.

17. If the tangent and normal at a point P of a conic meet the axis in the points T and G, and PN be the ordinate to the axis, then TN is called the 'Subtangent' and NG the 'Subnormal.'

N.B.—The following theorems may be postponed until the student has read the parabola.

PROP. 6.

18. *If* TPP′, TQQ′ *be two secants of a conic, then the ratio* TP . TP′ : TQ . TQ′ *is constant so long as the inclinations of the secants to the axis remain unaltered.*

For in the figure of [Prop. 3] let P′PT cut the circle in L and L′; then

$$\mathrm{TP} : \mathrm{SE} :: \mathrm{TZ} : \mathrm{ZE}$$
$$\text{and } \mathrm{TP}' : \mathrm{SE}' :: \mathrm{TZ} : \mathrm{ZE}'$$
$$\therefore\ \mathrm{TP} . \mathrm{TP}' : \mathrm{SE} . \mathrm{SE}' :: \mathrm{TZ}^2 : \mathrm{ZE} . \mathrm{ZE}'$$

and $\mathrm{ZE} . \mathrm{ZE}' = \mathrm{ZL} . \mathrm{ZL}' = \mathrm{TZ}^2 \sim \mathrm{TL}^2$. [Euc. ii. 5 Cor.]

Now the ratio of TL to Tx is constant for all positions of T, and the ratio of TZ to Tx is constant so long as the *angle* ZTx remains unaltered.

Hence the ratio of TL to TZ remains unaltered.

$$\therefore\ \mathrm{TL}^2 : \mathrm{TZ}^2 = \text{constant}$$
$$\therefore\ \mathrm{TZ}^2 : \mathrm{TZ}^2 \sim \mathrm{TL}^2 = \text{constant}$$

Hence $\mathrm{TP} . \mathrm{TP}' : \mathrm{SE} . \mathrm{SE}' = \text{constant}$ for all positions of T.

In the same way if SFF′ be the secant of the auxiliary circle, corresponding to TQQ′,

$$\mathrm{TQ} . \mathrm{TQ}' : \mathrm{SF} . \mathrm{SF}' = \text{constant}$$

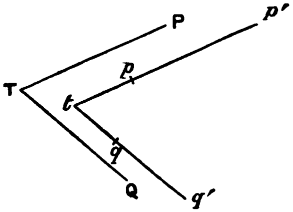
Hence $\mathrm{TP} . \mathrm{TP}' : \mathrm{TQ} . \mathrm{TQ}' = \text{constant}$ for all positions of T.

Q.E.D.

19. This is the most general, and consequently the most important, theorem in the whole theory of conics, and nearly all the other propositions in conics will be particular cases of this, or may be easily deduced from it. The student will find that it holds good in every conceivable position of the point T, and for all directions of the lines TPP′ and TQQ′. In order to familiarise the reader with the meaning of the theorem we shall point out several particular cases by way of illustration.

20. Suppose the points P and P′, and also Q and Q′, are taken indefinitely near to each other, so that TP and TQ become a pair of tangents to the conic. Let *tpp'*, *tqq'* be a pair of secants parallel to them,

FIG. 8.



Then the theorem proves that

$$TP^2 : TQ^2 :: tp . tp' : tq . tq'$$

21. Again, if it be possible to draw another pair of tangents OR, OR' respectively parallel to TP and TQ, the theorem also proves that

$$TP^2 : TQ^2 :: OR^2 : OR'^2$$
$$\text{or } TP : TQ :: OR : OR'$$

22. *Cor.*—It is clear from [Prop. 4] that if T be a point on the axis TP is equal to TQ; hence

$$tp . tp' = tq . tq'$$

and a circle would circumscribe the quadrilateral *pp'q'q*.

EXAMPLES.

1. If *tpp'*, *tqq'* be equally inclined to the axis of a conic, then a circle will circumscribe the quadrilateral *pp'q'q*, and *p'q* and *pq'* are also equally inclined to the axis.

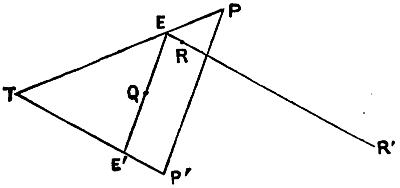
2. If through the points P and Q lines be drawn parallel to TQ and TP meeting the conic in P' and Q', then the line P'Q' is parallel to the line PQ.

PROP. 7.—THEOREM.

23. *If a tangent be drawn parallel to a chord of a conic, the portion of this tangent which is intercepted by the tangents at the ends of the chord is bisected at its point of contact.*

Let PP′ be the chord, TP, TP′ the tangents at its ends, and EQE′ the tangent parallel to PP′ touching the curve in the point Q. Through E draw a secant ERR′ parallel to the tangent TP′.

FIG. 9.



Then $$ER . ER' : EP^2 :: TP'^2 : TP^2 \qquad \text{[Art. 20.]}$$
$$:: E'P'^2 : EP^2$$
$$\therefore\ ER . ER' = E'P'^2$$

Again $$EQ^2 : ER . ER' :: E'Q^2 : E'P'^2$$

Hence $$EQ = E'Q \qquad \text{Q.E.D.}$$

Ex. If a line be drawn through E′ meeting the curve in r and r', and intersecting ERR′ in the point O, prove that

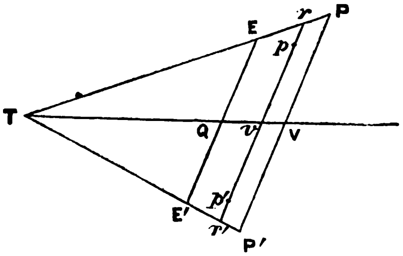
$$ER . ER' : OR . OR' :: E'r . E'r' : Or . Or'$$

PROP. 8.—THEOREM.

24. *The middle points of a series of parallel chords in a conic all lie in one straight line which passes through the intersection of the tangents at the extremities of any one of the chords.*

Let PP′ be one of the chords TP, and TP′ the tangents at its ends, EQE′ the tangent parallel to it.

FIG. 10.



Then, since EQ=E′Q, if TQ be produced to meet PP′ in the point V, PV=P′V, and if any other chord pp' parallel to PP′ meet TP, TV, and TP′ in the points r, v, and r' respectively, rv is also equal to $r'v$.

Again $$rP^2 : EP^2 :: r'P'^2 : E'P'^2$$

Hence $$rp\,.\,rp' : EQ^2 :: r'p'\,.\,r'p : E'Q^2$$
$$rp\,.\,rp' = r'p'\,.\,r'p$$

or $$rv^2 - pv^2 = r'v^2 - p'v^2$$
$$\therefore\ pv = p'v$$

In the same way the middle point of any other chord parallel to PP′ lies on the straight line QV. Q.E.D.

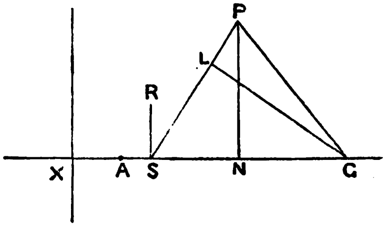
PROP. 9.—THEOREM.

25. *If* G *be the foot of the normal at a point* P *of a conic, and* GL *be drawn perpendicular to the focal radius* SP, *then* PL *is equal to the semi-latus-rectum.*

Let PN be the ordinate at the point P and SR the semi-latus-rectum.

Then, since the angles PLG and PNG are right angles, a circle will pass round PLNG.

FIG. 11.



$$\therefore\ SP \cdot SL = SG \cdot SN$$

$$\therefore\ SL : SN :: SG : SP$$

$$:: SA : AX$$

Again $\quad SP : NX :: SA : AX$

$$\therefore\ PL : SX :: SA : AX$$

for $\quad PL = SP - SL$

and $\quad SX = NX - SX$

But $\quad SR : SX :: SA : AX$

$$\therefore\ PL = SR$$

Q.E.D.

PROP. 10.—*THEOREM.*

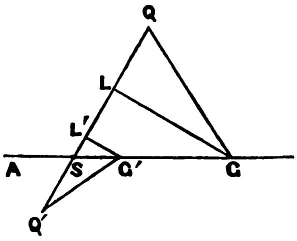
26. *If* QSQ′ *be any focal chord of a conic and* RSR′ *the latus rectum, then*

$$2SQ \cdot SQ' = RS \cdot QQ'$$

or, in other words, *the semi-latus rectum is a harmonic mean between the segments of any focal chord.*

Let the normals at Q and Q' meet the axis in the points G and G'; draw GL and $G'L'$ perpendicular to QQ'.

FIG. 12.



Then QL and $Q'L'$ are each equal to SR

Now $SL : SL' :: SG : SG'$

$:: SQ : SQ'$ [Prop. 5.]

$\therefore\ SQ - SR : SR - SQ' :: SQ : SQ'$

or SR is a harmonic mean between SQ and SQ'

and $(SQ - SR)SQ' = SQ(SR - SQ')$

$\therefore\ 2SQ \, . \, SQ' = SR(SQ + SQ')$

$= SR \, . \, QQ'$ Q.E.D.

27. *Cor.*—If any two secants be drawn through a point, the ratio of the rectangles contained by their segments is equal to the ratio of the rectangles contained by the segments of the focal chords parallel to them [Prop. 6], and therefore equal to the ratio of the lengths of those chords.

CHAPTER II.

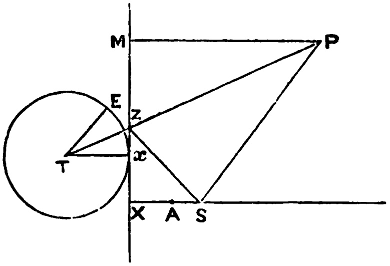
THE PARABOLA.

Def.—The curve traced out by a point which moves in such a manner that its distance from a given point is always equal to its distance from a given straight line is called a 'Parabola.'

TRACING THE CURVE.

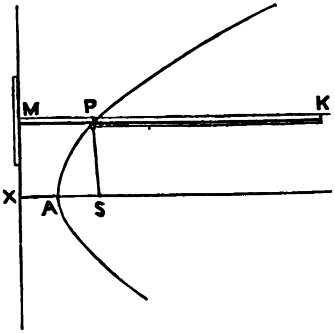
28. Draw the auxiliary circle of any point T on the opposite side of the directrix to the focus. Let SE and TP be corresponding tangents to this circle and the parabola respectively.

FIG. 13.



Suppose the centre T to move upwards from the axis, keeping the same distance from the directrix, and suppose the tangents SE and TP to alter accordingly. Then the point E approaches the point x, and the angles ETZ and ZTx continually diminish; hence the angles SPZ and ZPM also gradually diminish; therefore the sides SP, PZ, and PM gradually increase. We therefore come to this conclusion with regard to the parabola: Starting from the point A, it gradually recedes both from the focus and directrix; and, since the inclination of the tangent TP to the axis gradually diminishes, the curve is concave to the axis. Again, since the point E will never coincide with the point x, the tangent TZ can never become parallel to the axis. It is easy to see that the auxiliary circle of any other point on the tangent TP touches the line SE; and hence we have traced the whole curve,

FIG. 14.



which is of the form in the figure. It lies wholly on one side of the directrix. It has one branch which extends to an infinite distance. It is the same above the axis as it is below.

29. *Mechanical Construction of the Curve.*

Let MK represent a T square capable of sliding along a rigid directrix MX. A string KPS whose length is equal to MK is fastened at K and S, and passes through a ring P which can slide along the T square. As the T square moves upwards it is clear that SP always equals PM, and a pencil at the ring P traces out a parabola.

PROP. 1.—*THEOREM.*

30. *The tangent at any point* P *bisects the angle between the focal distance* SP *and the perpendicular* PM *upon the directrix.*

For, using the figure and construction of Art. 28,

$$SPZ = ETZ$$
$$MPZ = xTZ$$
$$\text{but} \quad ETZ = xTZ$$
$$\therefore SPZ = MPZ$$

31. *Cor.* 1.—Hence it is clear that the tangent at the vertex is at right angles to the axis, for it makes equal angles with SA and AX.

Cor. 2.—Since $EZT = xZT$

$\therefore$ PZ also bisects the angle SZM

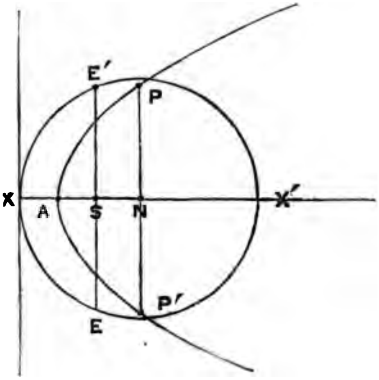
PROP. 2.—THEOREM.

32. *If* PN *be an ordinate to the axis of a parabola,*

$$PN^2 = 4AS . AN$$

Draw the auxiliary circle of the point N touching the directrix in the point X, and cutting the axis in the point X′ and the latus rectum in E and E′.

FIG. 15.



Then as in [Art. 5] PN=SE

$$\begin{aligned}SX' &= SN+NX\\ &= SN+(SN+SX)\\ &= 2SN+2AS\\ &= 2AN\end{aligned}$$

Hence $PN^2=SE^2=SX.SX'=2AS.2AN$

$$\therefore\ PN^2=4AS.AN \qquad \text{Q.E.D.}$$

PROP. 3.—THEOREM.

33. *If* TP, TP′ *be a pair of tangents to a parabola, and* Tx, PM, *and* P′M′ *be the perpendiculars on the directrix, then* Mx=M′x

In the figure of Art. 28 we have

$$Zx=ZE$$

and $$ZM=ZS$$

Therefore $$Mx=SE$$

In the same way if SE′ be the tangent to the auxiliary circle which corresponds to TP′

$$M'x=SE'$$

Hence $$Mx=M'x \qquad \text{Q.E.D.}$$

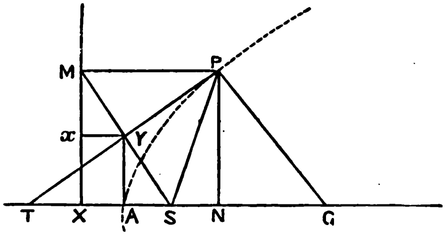
PROP. 4.—THEOREM.

34. *If the tangent and normal at any point* P *meet the axis in* T *and* G, *and* PN *be the ordinate,*

AT=AN, SP=ST, *and* NG=2AS

Let the tangent at P meet the tangent at the vertex in Y, and draw PM and Yx perpendicular to the directrix.

FIG. 16.



Then by the last article Mx is equal to Xx.

Therefore PY is equal to TY

Hence AN is equal to AT

Again STY=TPM=SPY

$\therefore$ SP=ST

Also $PN^2=TN.NG$

Therefore $4AS.AN=TN.NG$

$=2AN.NG$

Or NG=2AS Q.E.D.

PROP. 5.—*THEOREM.*

35. *If the tangent at the vertex meet the tangent at* P *in* Y, *then* SY *is perpendicular to the tangent and*

$$SY^2=AS.SP$$

Draw Yx and PM perpendicular to the directrix; then, since

Mx=Xx [Prop. 3]

$\therefore$ PY=TY

Again SP=ST [Prop. 4]

and the side SY is common to the two triangles

$$\therefore\ \text{SYT}=\text{SYP}$$

$$\therefore\ \text{SY is perpendicular to TP}$$

Next, since TYS is a right angle,

$$\therefore\ \text{SY}^2=\text{SA}\,.\,\text{ST}$$
$$=\text{SA}\,.\,\text{SP} \qquad \text{Q.E.D.}$$

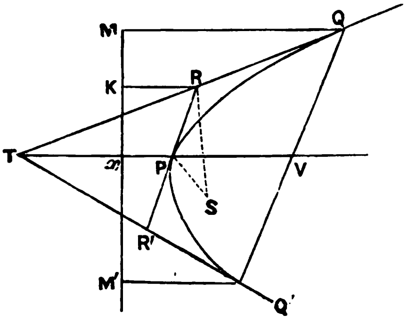
Def.—A straight line drawn through any point on a parabola parallel to the axis is called a diameter.

PROP. 6.—THEOREM.

36. *A diameter bisects all chords parallel to the tangent at its extremity.*

Let QQ′ be any chord, and let the tangents at Q and Q′ intersect in T. Draw Tx, QM, Q′M′ perpendicular to the directrix.

FIG. 17.



Then Mx=M′x [33]. Therefore if Tx be produced it will bisect QQ′. Let it meet QQ′ in V and the curve in P. Let the tangent at P meet TQ, TQ′ in R, R′, and let RK be the perpendicular upon the directrix. Then MK is equal to Kx; hence also TR is equal to RQ. In the same way TR′ is equal to R′Q′. Hence RR′ is parallel to QQ′. Therefore the diameter Px bisects all chords parallel to the tangent at P, and the tangents at the extremity of any chord intersect on the diameter which bisects that chord. Q.E.D.

37. *Cor.*—Since TR is equal to RQ and QV=Q′V

$$\therefore \quad TP=PV$$
$$\text{and} \quad RP=R'P$$

Def.—The straight line QV drawn through any point Q on a parabola parallel to the tangent at P and meeting the diameter through P in the point V is called an 'Ordinate' to that diameter.

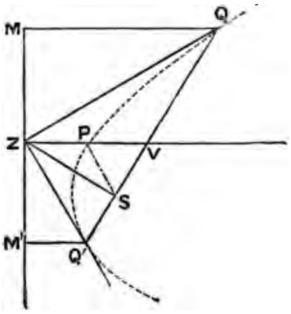
The double ordinate of a diameter which passes through the focus is called its 'Parameter.'

PROP. 7.—THEOREM.

38. *The tangents at the extremities of a focal chord intersect at right angles in the directrix.*

Let QQ′ be the focal chord. The tangents at its extremities intersect the directrix on the perpendicular SZ to the chord by Art. 12, Ex. 1. Let QM, Q′M′ be the perpendiculars upon the directrix.

FIG. 18.



Then QZ and Q′Z bisect the angles SZM and SZM′. [Art. 31.]

Hence QZQ′ is a right angle. Q.E.D.

PROP. 8.—*THEOREM.*

39. *The parameter of a diameter* PV *is equal to* 4SP.

Let QQ′ be the parameter; let the tangents at its extremities intersect the directrix in Z: then ZP is the diameter which bisects QQ′. Let it meet it in V; then

$$SQ=QM \text{ and } SQ'=Q'M'$$

$$\begin{aligned}\therefore\ QQ' &= QM+Q'M'\\ &= 2VZ\\ &= 4PZ\\ &= 4SP\end{aligned}$$

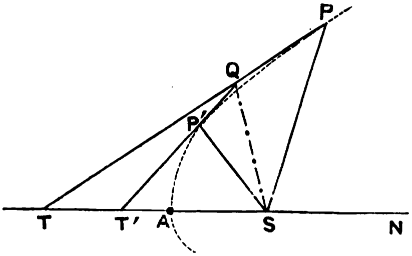
Q.E.D.

PROP. 9.—*THEOREM.*

40. *If* QP, QP′ *be a pair of tangents to a parabola, the external angle between them is equal to half that between* SP *and* SP′.

Let the tangents meet the axis in T and T′, and join PS, QS, P′S.

FIG. 19.



Then, as in Art. 35, the angle SPT is equal to STP.

∴ the exterior angle PSN is double of the angle STP.

Similarly the angle P′SN is double of the angle ST′P′.

Hence the difference PSP′ is double of the difference TQT′.

Q.E.D.

PROP. 10.—*THEOREM.*

41. *If* QP, QP′ *be a pair of tangents to a parabola, the triangles* SQP *and* SQP′ *are similar, and*

$$SP \,.\, SP' = SQ^2$$

For the angle PSP′ is double of TQT′ and QS bisects it. [Art. 15.]

Hence each of the angles PSQ and P′SQ is equal to TQT′.

If each of these equal angles be taken from the angle SQT the remainders must be equal.

Therefore the angle SQP′ is equal to the angle SPQ.

Therefore the triangles SPQ and SQP′ are similar.

Hence also $SP : SQ :: SQ : SP'$

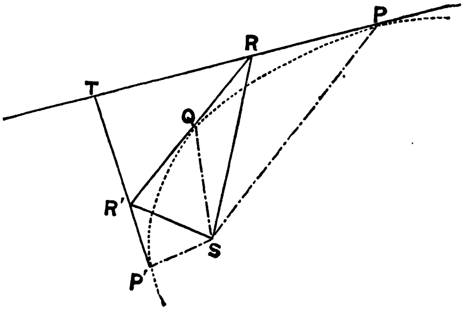
or $SP \,.\, SP' = SQ^2$ Q.E.D.

PROP. 11.—*THEOREM.*

42. *The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.*

Let TRR′ be the triangle; P, Q, P′ the points of contact of the tangents. Join all these points to the focus.

FIG. 20.



Then since the lines RS and R′S bisect the angles PSQ and QSP′,

Therefore the angle PSP′ is double of the angle RSR′.

But the angle PSP′ is also double of the external angle between TP and TP′. [Art. 40.]

Therefore this external angle is equal to the angle RSR′.

Hence the angles RSR′ and RTR′ are together equal to two right angles.

Therefore a circle will go round the quadrilateral TRSR′.

Q.E.D.

PROP 12.—*THEOREM.*

43. *If* QVQ′ *be a double ordinate to a diameter* PV, *then*

$$QV^2 = 4SP.PV$$

Let the tangents at Q and Q′ meet the diameter in the point T [Art. 36]; join PS, RS, QS.

Then because the tangent at P bisects the angle between SP and Px [Art. 30],

Therefore the angle SPR is equal to the angle RPT.

Again, the angle QSP is double of the angle TRP. [Art. 40.]

And RS bisects it. [Art. 15.]

Therefore the angle RSP is equal to the angle TRP.

Hence the triangles SPR and TPR are similar.

Therefore $SP : PR :: PR : PT$

Or $RP^2 = SP.PT$

Now QV is double of RP

And TP is equal to PV

Hence $QV^2 = 4SP.PV$ Q.E.D.

44. N.B.—Since 4SP is the parameter of the diameter PV, the above is expressed by saying that QV is a mean proportional between PV and the parameter of PV.

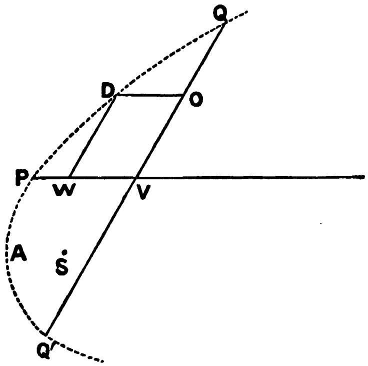
The following articles (45 and 46) have been proved generally for all conics in Art. 26, but for the particular case of the parabola the proof adjoined will be found very easy.

PROP. 13.—*THEOREM.*

45. *If* OQQ′, ORR′ *be any two secants of a parabola, then the ratio of the rectangles* OQ . OQ′ *and* OR . OR′ *is the same for all parallel chords.*

Let the diameter through O meet the curve in D, and let PV be the diameter which bisects QQ′. Draw the ordinate DW to the diameter PV.

FIG. 21.



Then $\quad OQ . OQ' = QV^2 - OV^2 = QV^2 - DW^2$

And $\quad QV^2 = 4SP . PV$ and $DW^2 = 4SP . PW$

Hence
$$OQ . OQ' = 4SP . VW$$
$$= 4SP . DO$$

Similarly $\quad OR . OR' = 4SP' . DO$

Therefore $\quad OQ . OQ' : OR . OR' :: 4SP : 4SP'$

Now 4SP and 4SP′ are the lengths of the focal chords parallel to QQ′ and RR′, or the parameters of the diameters which bisect them. Hence the ratio of these rectangles only depends on the directions of the secants. Q.E.D.

46. *Cor.*—If QSQ′ be a focal chord
$$SQ . SQ' = 4RS . AS$$
$$= QQ' . AS$$

CHAPTER III.

THE ELLIPSE.

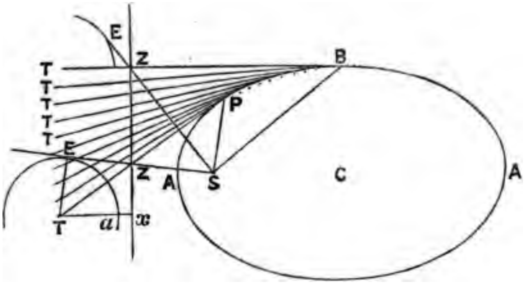
DEFINITION.

The curve traced out by a point which moves in such a manner that its distance from a given point is in a constant ratio, *less than unity*, to its distance from a given straight line is called an Ellipse.

TRACING THE CURVE.

47. In SX take a point A such that the ratio SA : AX =the given ratio; then A is evidently a point in the curve.

FIG. 22.



Draw the auxiliary circle at any point T on the side of the directrix remote from the focus. Let SE, TP be corresponding tangents to the circle and ellispe respectively. Since the eccentricity is less than unity Ta is less than Tx; ∴ the auxiliary circle does not cut the directrix; ∴ Z always lies between E and S. Therefore T and P will clearly lie on opposite sides of the directrix. If T be taken on the same side of the directrix as the focus, it will be equally clear on drawing the figure that T and P lie on the same side of the directrix. Hence the curve lies wholly on one side of the directrix.

By taking successive positions of T and drawing the tangents we arrive at the following conclusions:—

1. Starting from A, the curve is concave to the axis.
2. It recedes from the axis until we arrive at a point B where the tangent becomes parallel to the axis, after which it again approaches the axis and cuts it at the point A′, where

$$SA' : A'X :: SA : AX$$

3. The axis divides the curve into two equal and similar parts.

DEFINITIONS.

1. The line AA′ is called the 'Transverse Axis' of the ellipse.

2. The middle point of the axis is called the 'Centre.'

3. The double ordinate through the centre is called the 'Conjugate Axis.'

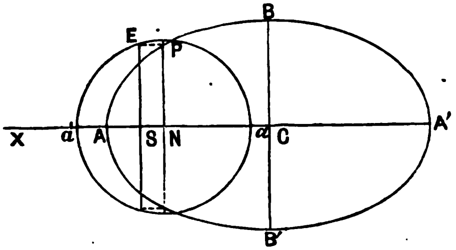
PROP. 1.—*THEOREM.*

48. *If* PN *be an ordinate to the axis,* BB′ *the conjugate axis, and* C *the centre,* then

$$PN^2 : AN . NA' :: BC^2 : AC^2$$

Draw the auxiliary circle of N, cutting the axis in *a, a′* and the latus rectum in E.

FIG. 23.



Then as in Art. 5 $\quad PN = SE$

Now $\quad Na : NX :: SA : AX \qquad$ [Def. 11]

$$\therefore\ Xa : NX :: SX : AX$$

$$\therefore\ Xa - SX : NX - AX :: SX : AX$$

$$\therefore\ Sa : AN :: SX : AX$$

Similarly $\quad Sa' : A'N :: SX : A'X$

$$\therefore\ Sa \,.\, Sa' : AN \,.\, NA' : SX^2 : AX \,.\, XA'$$

But $\quad Sa \,.\, Sa' = SE^2 = PN^2$

$$\therefore\ PN^2 : AN \,.\, NA' :: SX^2 : AX \,.\, XA'$$

Taking the particular case where PN coincides with the conjugate axis

$$BC^2 : AC \,.\, CA' :: SX^2 : AX \,.\, XA'$$

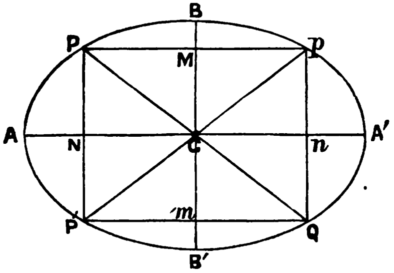
Hence $\quad PN^2 : AN \,.\, NA' :: BC^2 : AC^2 \qquad$ Q.E.D.

49. It will be observed that the last article is a special case of the general theorem, Art. 18. It may also be written

$$PN^2 : CA^2 - CN^2 :: BC^2 : AC^2$$

50. *On the Symmetry of the Ellipse.*

FIG. 24.



If Cn be taken equal to CN, we see that $A'n = AN$ and $A'N = An$. Hence [Art. 48] pn is equal to PN and PM to pM. Again, if pn meet the curve again in Q, we have $pn = Qn$. Hence PCQ is a straight line and is bisected at C. Also we see that the curve is symmetrical with respect to either axis.

51. *Cor.* $PM^2 : BM . B'M :: AC^2 : BC^2$

For $PN^2 : AC^2-CN^2 :: BC^2 : AC^2$

$\therefore\ CM^2 : AC^2-PM^2 :: BC^2 : AC^2$

$\therefore\ BC^2-CM^2 : PM^2 :: BC^2 : AC^2$

Hence $PM^2 : BM . MB' :: AC^2 : BC^2$

Def.—Any line PCQ through the centre is called a Diameter.

52. Conversely, if we have given that $PN^2 : AN . A'N$ in a constant ratio, the existence of a focus and directrix may be established.

For, taking the particular case where PN coincides with BC, we get

$$PN^2 : AN . A'N :: BC^2 : AC^2$$

Again, take a point S such that $CS^2 = AC^2 - BC^2$, and also a point X such that $CS . CX = CA^2$. Then if we describe on AA′ a circle and draw SL perpendicular to AA′ meeting the circle in L, XL is the tangent to the circle and SL = BC.

$$[\text{For } SL^2 = AS . SA' = CA^2 - CS^2 = CB^2]$$

and $\therefore\ BC^2 : AC^2 :: SL^2 : CL^2$

$:: SX^2 : LX^2$

$:: SX^2 : AX . A'X$

and as before $Sa . Sa' : AN . A'N :: SX^2 : AX . A'X$

Hence $SE = PN$

Hence $SP = NE$

But $NE : NX :: SA : AX$

$\therefore\ SP : NX :: SA : AX$

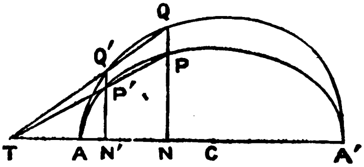
A similar method will apply to all conics.

PROP. 2.—*THEOREM.*

53. *If the ordinate* PN *meet the circle on the transverse axis as diameter in* Q,

$$PN : QN :: BC : AC$$

Fig. 25.



Because $PN^2 : AN . NA' :: BC^2 : AC^2$

and $AN . NA' = QN^2$ [Euc. VI. 8 Cor.]

$$\therefore PN^2 : QN^2 :: BC^2 : AC^2$$
$$\therefore PN : QN :: BC : AC$$ Q.E.D.

54. *Cor.* 1.—If $Q'P'N'$ be any other ordinate,

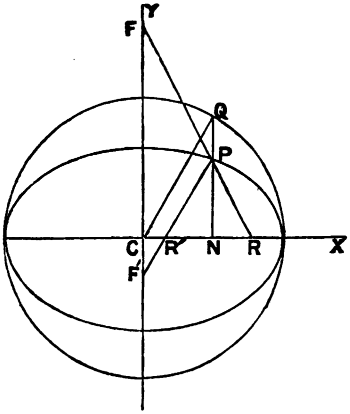
$$PN : QN :: P'N' : Q'N'$$

Hence it is evident that PP', QQ' intersect on the axis.

55. *Cor.* 2.—If P' be indefinitely near to P, Q' will also be indefinitely near to Q. Hence tangents at P and Q intersect on the axis.

56. We can hence deduce a mechanical construction for the ellipse.

FIG. 26.



Let CX, CY be two fixed rods at right angles, FR a rod of given length sliding between them, which is divided at P in a given ratio; then the point P will trace out an ellipse of which PF, PR are the lengths of the major and minor semi-axes respectively.

For if $PR'F'$ be drawn equally inclined to CX with PR, then $PF = PF'$ and $PR = PR'$. Let PN be the ordinate through P, and CQ parallel to PF' meeting PN in Q; then $CQ = PF' = PF$ and is constant, and $\therefore$ the locus of Q is a circle.

Also

$$\begin{aligned} PN : QN &:: PR' : QC \\ &:: PR : PF \\ &:: \text{a constant ratio} \end{aligned}$$

$\therefore$ the locus of P is an ellipse which clearly has PF and PR for its semi-major and semi-minor axes respectively.

It is thus clear that every point of the above rod traces out an ellipse, and that its middle point traces out a circle.

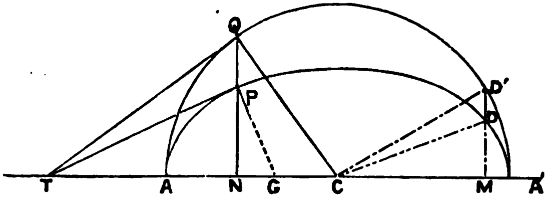
PROP. 3.—*THEOREM.*

57. *If the tangent at* P *meet the axis in* T, *and* PN *be the ordinate,*

$$CN . CT = CA^2$$

Let PN meet the circle on the axis as diameter in Q. Join CQ, TQ.

Fig. 27.



Then TQ is a tangent to the circle. [Art. 55.]

$\therefore$ TQC is a right angle

$\therefore$ $CN . CT = CQ^2$ [Euc. VI. 8 Cor.]

$= CA^2$ Q.E.D.

58. Note $CN . NT = QN^2 = AN . NA'$

59. If the tangent at P meet the conjugate axis in *t*, and PM be the ordinate to that axis,

$$ct : PN :: CT : NT$$

$$\therefore ct . CM : PN^2 :: CT . CN : CN . NT$$

or

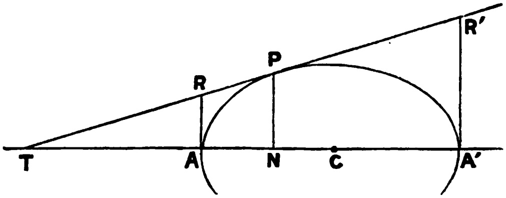
$$\begin{aligned} ct . CM : CA^2 &:: PN^2 : AN . NA' \\ &:: BC^2 : AC^2 \end{aligned}$$

$$\therefore ct . CM = BC^2.$$

N.B.—These are only simple cases of Prop. 9.

60. We have been led to the result in Art. 57 by continuous deduction from the primary Prop. in Art. 18; but it may be obtained more directly in the following manner. Let the tangent at P meet those at A and A′ in R and R′ respectively; then

Fig. 28.



since RA, RP are respectively parallel to R′A′, R′P

$$\therefore \text{RA} : \text{R}'\text{A}' :: \text{RP} : \text{R}'\text{P} \qquad \text{[Art. 21.]}$$

Hence $$\text{TA} : \text{TA}' :: \text{AN} : \text{NA}'$$

or $$\text{CT}-\text{CA} : \text{CT}+\text{CA} :: \text{CA}-\text{CN} : \text{CA}+\text{CN}$$

$$\therefore \text{CT} : \text{CA} :: \text{CA} : \text{CN}$$

$$\therefore \text{CT}\,.\,\text{CN} = \text{CA}^2 \qquad \text{Q.E.D.}$$

PROP. 4.—THEOREM.

61. *If the normal at the point* P *meet the axis in* G, *and* PN *be the ordinate,*

$$\text{NG} : \text{CN} :: \text{BC}^2 : \text{AC}^2$$

For, using the figure and construction of Prop. 3, since TPG and TQC are right angles,

$$\text{PN}^2 = \text{TN}\,.\,\text{NG} \text{ and } \text{QN}^2 = \text{TN}\,.\,\text{NC}$$

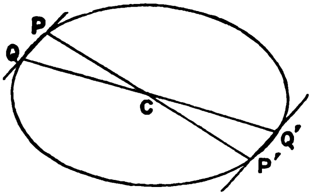
But $$\text{PN}^2 : \text{QN}^2 :: \text{BC}^2 : \text{AC}^2$$

$$\therefore \text{NG} : \text{CN} :: \text{BC}^2 : \text{AC}^2 \qquad \text{Q.E.D.}$$

PROP. 5.—THEOREM.

62. *If* PCP′, QCQ′ *be any two diameters, then* PP′ *shall be parallel to* QQ′.

FIG. 29.



In the triangles PCQ, P′CQ′ we have

PC, QC=P′C, Q′C each to each

and ∠PCQ=∠P′CQ′

∴ ∠CPQ=∠CP′Q′

∴ PQ is parallel to P′Q′ Q.E.D.

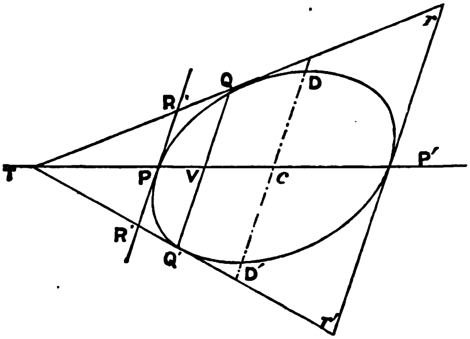
63. *Cor.* 1.—If P be taken indefinitely near to Q, then Q′ will be indefinitely near to P′, and PQ, P′Q′ produced will become tangents. Hence the tangents at the extremities of a diameter are parallel.

PROP. 6.—THEOREM.

64. *A diameter bisects all chords parallel to the tangents at its extremities.*

Let QQ′ be a chord parallel to the tangents at P and P′. Let the tangents at Q, Q′ meet those at P, P′ in R, R′ and r, r' respectively, and intersect in T. Let QQ′ meet PP′ in V.

FIG. 30.



Then RR' and rr' are parallel [Art. 63] and are bisected at P and P' respectively. [Art. 23.]

Hence PP' bisects QQ' in V and passes through T.

In the same way any other chord parallel to the tangents at P and P' is bisected by PP'. Q.E.D.

DEFINITION.

QV and $Q'V$ drawn parallel to the tangents at P and P' are called ordinates to the diameter PCP'.

PROP. 7.—THEOREM.

65. *If* QVQ' *be a double ordinate of the diameter* PCP', *and if the diameter* DCD' *be parallel to* QVQ', *then shall*

$$QV^2 : PV \cdot VP' :: CD^2 : CP^2$$

For, using the figure of Art. 64,

$$QV \cdot VQ' : PV \cdot VP' :: CD \cdot CD' : CP \cdot CP' \quad \text{[Art. 18.]}$$

But $QV = VQ'$, $CD = CD'$, and $CP = CP'$

$$\therefore\ QV^2 : PV \cdot VP' :: CD^2 : CP^2 \qquad \text{Q.E.D.}$$

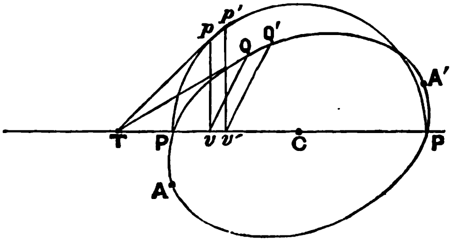
Cor.—This may be written

$$QV^2 : CP^2 - CV^2 :: CD^2 : CP^2$$

PROP. 8.—*THEOREM.*

66. *If* Qv *be an ordinate to the diameter* PCP′, *and the perpendicular* vp *be drawn meeting the circle on* PP′ *as diameter in* p, *then the tangents at* Q *and* p *meet on the diameter* P′P *produced.*

Fig. 31.



For let Q′v' be another ordinate, and let Q′Q meet PP′ in T; join Tp and let it meet the perpendicular through v' in p'. Then, as in Prop. 2, it may be shown that

$$Qv : pv' :: CD : CP$$

Again $$pv : p'v' :: Tv : Tv'$$
$$:: Qv : Q'v'$$

Hence $$Q'v' : p'v' :: CD : CP$$

$\therefore$ p' also lies on the circle

Hence the chords QQ′, pp' meet the diameter in the same point T, and, as in Art. 55, the tangents at Q and p meet on the diameter PP′.

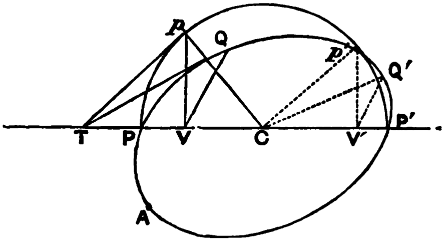
PROP. 9.—*THEOREM.*

67. *If the tangent at* Q *meet the diameter* PP′ *in* T, *and* QV *be the ordinate to that diameter,*

$$CV . CT = CP^2$$

Through V draw Vp perpendicular to PP' to meet the circle on PP' as diameter in p. Join Tp, pC.

FIG. 32.



Then Tp is a tangent to the circle

$\therefore$ TpC is a right angle

$\therefore$ $CV \,.\, CT = Cp^2$ [Euc. vi. 8. Cor.]

$= CP^2$ Q.E.D.

By using the figure of Art. 64, this may also be proved more directly, as in the case of the corresponding prop. in Art. 60.

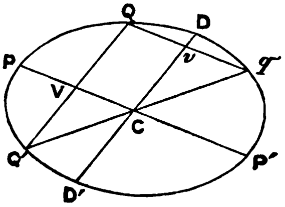
Note. $CV \,.\, VT = pV^2 = PV \,.\, VP'$

PROP. 10.—*THEOREM.*

68. *If a diameter* PCP' *bisect chords parallel to another diameter* DCD', *then shall* DCD' *bisect chords parallel to* PCP'.

Draw QQ' parallel to CD meeting CP in V. Draw the diameter $Q'Cq'$. Join Qq' meeting CD in v.

FIG. 33.



Because $QV = Q'V$ and $CQ' = Cq'$

$\therefore$ Qq' is parallel to CP

Again since CD is parallel to QQ' and $q'C = CQ'$

$\therefore$ $q'v = Qv$

$\therefore$ CD bisects chords parallel to CP Q.E.D.

Def.—A diameter which bisects chords parallel to another is said to be conjugate to it.

A pair of chords QQ', Qq' drawn from any point Q to the extremities of a diameter are called 'Supplemental Chords,' and we see that they are parallel to a pair of conjugate diameters.

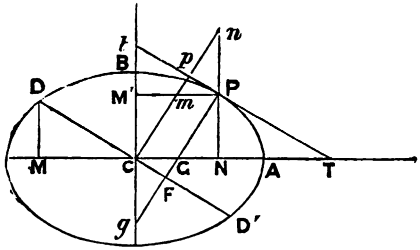
PROP. 11.—*THEOREM.*

69. *If the normal at a point* P *meet the axes in* G *and* g, *and the diameter conjugate to* CP *in* F, *then*

$$PF . PG = BC^2 \text{ and } PF . Pg = AC^2$$

Let the tangent at P meet the axes in T and t. Draw Cp perpendicular to the tangent meeting the ordinates PM′, PN in m and n.

FIG. 34.



Because $Cm = PG$, $Cn = Pg$, and $Cp = PF$

$\therefore$ $PF . PG = Cp . Cm$

$= Ct . CM'$ [Euc. iii. 36. Cor.]

$= BC^2$ [Art. 59.]

Again $$PF . Pg = Cp . Cn$$
$$= CT . CN \quad \text{[Euc. iii. 36 Cor.]}$$
$$= AC^2 \quad \text{[Art. 57.]}$$

Q.E.D.

PROP. 12.—*THEOREM.*

70. *If* CP, CD *be a pair of conjugate semidiameters, and the ordinates* PN, DM *be produced to meet the circle on* AA′ *as diameter in* Q *and* D′, *then the angle* QCD′ *shall be a right angle.*

$$PN : CM :: BC : AC$$

and $$CM^2 = AN . NA'$$

For, using the figure and construction of Art. 57,

Since TP is parallel to CD, the triangles TNP and CMD are similar.

$$\therefore\ TN : CM :: PN : DM$$
$$:: QN : D'M \quad \text{[Art. 53.]}$$

Hence TNQ and CMD′ are similar.

$$\therefore\ TQ \text{ is parallel to } CD'$$

$$\therefore\ QCD' = CQT = \text{a right angle} \ldots . \text{[A]}$$

Again, in the right-angled triangles QCN, D′CM

$$CQN = D'CM$$

because they are each the complement of QCN and the side CQ=the side CD′.

∴ the triangles are altogether equal.

$$\therefore\ CM = QN \text{ and } D'M = CN$$

Now $$PN : QN :: BC : AC$$

$$\left.\begin{array}{ll} & \therefore\ PN : CM :: BC : AC \\ \text{Similarly} & DM : CN :: BC : AC \end{array}\right\} \ldots . \text{[B]}$$

$$\left.\begin{array}{ll} \text{Thirdly} & CM^2 = QN^2 = AN . NA' \\ \text{Similarly} & CN^2 = AM . MA' \end{array}\right\} \ldots . \text{[C]}$$

N.B.—The results [A], [B], [C] ought to be remembered.

PROP. 13.—*THEOREM.*

71. *If* CQ, CQ′ *be a pair of conjugate semidiameters and* QV, Q′V′ *ordinates to any diameter* PCP′, *and if* Vp, V′p' *be drawn perpendicular to* PP′ *to meet the circle on* PP′ *as diameter in* p, p', *then the angle* pcp' *shall be a right angle*

$$QV : CV' :: CD : CP$$

and

$$CV'^2 = PV \, . \, VP'$$

This proposition is the same as Prop. 12, but has the ordinates oblique, and by using figure of Prop. 9 may be proved in exactly the same way.

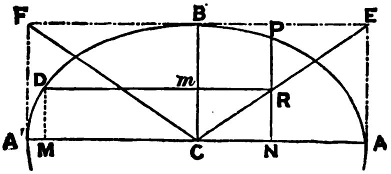
Def.—The diagonals CE, CF of the rectangles formed by the axes, and also parallel to the supplemental chords A′B and AB, are equal and conjugate diameters, and are called the ‘equiconjugate diameters.’

PROP. 14.—*THEOREM.*

72. *If an ordinate* PN *meet an equiconjugate diameter in* R, *then*

$$PN^2 + RN^2 = BC^2$$

FIG. 35.



For

$$PN^2 : CA^2 - CN^2 :: BC^2 : AC^2$$

and

$$RN^2 : CN^2 :: BC^2 : AC^2$$

$$\therefore \; PN^2 + RN^2 : CA^2 :: BC^2 : CA^2$$

$$\therefore \; PN^2 + RN^2 = BC^2$$

Similarly $Dm^2 + Rm^2 = AC^2$ Q.E.D.

73. In the same way if AA′, BB′ be *any* pair of conjugate diameters, the tangents at A, A′ and B, B′ will form a parallelogram; and if PN, Dm be the ordinates to AA′ and BB′ meeting a diagonal in R, then

$$\mathrm{PN}^2+\mathrm{RN}^2=\mathrm{BC}^2$$
$$\mathrm{D}m^2+\mathrm{R}m^2=\mathrm{AC}^2$$

PROP. 15.—*THEOREM.*

74. *The sum of the squares of a pair of conjugate diameters is constant.*

For, using the figure and construction of Art. 72,

$$\mathrm{DM}=\mathrm{RN}$$
$$\therefore\ \mathrm{DM}:\mathrm{CN}::\mathrm{BC}:\mathrm{AC}$$

$\therefore$ CP and CD are conjugate diameters. [Prop. 12.]

From the results

$$\mathrm{PN}^2+\mathrm{RN}^2=\mathrm{BC}^2$$
$$\mathrm{D}m^2+\mathrm{R}m^2=\mathrm{AC}^2$$

we obtain

$$\left.\begin{array}{l}\mathrm{PN}^2+\mathrm{DM}^2=\mathrm{BC}^2\\ \mathrm{CN}^2+\mathrm{CM}^2=\mathrm{AC}^2\end{array}\right\}\ \ldots\ [\mathrm{A}]$$

$\therefore$ by addition

$$\mathrm{CP}^2+\mathrm{CD}^2=\mathrm{AC}^2+\mathrm{BC}^2 \qquad \text{Q.E.D.}$$

N.B.—The results [A] are often useful.

75. *Cor.*—The lengths of the equiconjugate diameters are obtained by putting CP=CD

$$\therefore\ 2\mathrm{CP}^2=\mathrm{AC}^2+\mathrm{BC}^2$$

PROP. 16.—*THEOREM.*

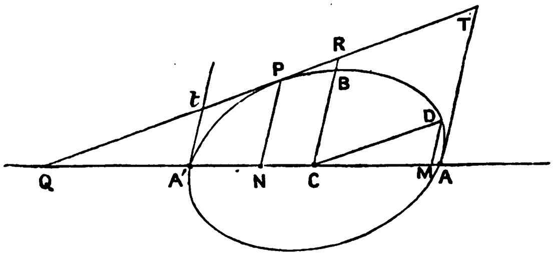
76. *If the tangent to an ellipse at a point* P *cut the tangents at the extremities of a diameter* ACA′ *in* T, t, *then shall*

$$\mathrm{PT}\,.\,\mathrm{P}t=\mathrm{CD}^2$$

CD *being the semidiameter parallel to* PT.

Let the tangent at P meet AA′ in Q. Draw the ordinates PN, DM to the diameter ACA′.

FIG. 36.



Then $$Pt : A'N :: CD : CM$$
$$PT : AN :: CD : CM$$
$$\therefore PT \,.\, Pt : AN \,.\, A'N :: CD^2 : CM^2$$

But $$CM^2 = AN \,.\, A'N \qquad \text{[Prop. 13]}$$
$$\therefore Pt \,.\, PT = CD^2 \;.\;.\;.\;.\; \text{[B]}$$

77. *Cor.*—Let CB be the semidiameter conjugate to CA meeting the tangent in R; then

$$tA' : tP :: BC : CD \qquad \text{[Art. 21]}$$
$$TA : TP :: BC : CD$$
$$\therefore tA' \,.\, TA : CD^2 :: BC^2 : CD^2$$
$$\therefore tA' \,.\, TA = BC^2 \;.\;.\;.\;.\; \text{[C]}$$

Again $$QP : QN :: CD : CM$$
$$PR : CN :: CD : CM$$
$$\therefore QP \,.\, PR : QN \,.\, CN :: CD^2 : CM^2$$

but $$CM^2 = AN \,.\, A'N = QN \,.\, CN \qquad \text{[Prop. 13]}$$
$$\therefore QP \,.\, PR = CD^2 \;.\;.\;.\;.\; \text{[D]}$$

N.B.—Since C*t*, CT bisect the supplemental chords A′P and AP they are conjugate diameters, and Prop. 16 may be enunciated thus:—

78. If the tangent to an ellipse at a point P meet a pair of conjugate diameters in T and *t*, then

$$Pt \,.\, PT = CD^2$$

It will then be observed, by producing TC to meet tA', that [C] and [D] are the same prop. as [B].

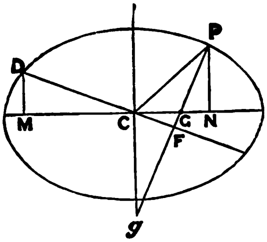
PROP. 17.—*THEOREM.*

79. *If* CP, CD *be conjugate diameters, and the normal at* P *meet* CD *in* F,

$$PF \cdot CD = AC \cdot BC$$

Draw the ordinates PN and DM. Let PF meet the axis in G.

FIG. 37.



Because PG, PN, NG are respectively perpendicular to CD, CM, DM

$\therefore$ PGN and CDM are similar triangles

$$\therefore PG : CD :: PN : CM$$
$$:: BC : AC \qquad \text{[Art. 70]}$$

$$\therefore PF \cdot PG : PF \cdot CD :: BC^2 : BC \cdot AC$$

But $\qquad PF \cdot PG = BC^2 \qquad$ [Art. 69]

$$\therefore PF \cdot CD = BC \cdot AC \qquad \text{Q.E.D.}$$

80. *Cor.* 1.—If PG be produced to meet the conjugate as in g,

$$Pg : CD :: CN : DM$$
$$:: AC : BC$$

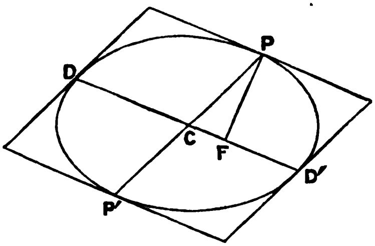
and above $\qquad PG : CD :: BC : AC$

$\therefore PG \cdot Pg = CD^2$ by combining the above ratios.

81. *Cor.* 2.—*The tangents at the extremities of a pair of conjugate diameters form a parallelogram whose area is constant.*

Let PCP′ and DCD′ be a pair of conjugate diameters.

Fig. 38.



The tangents at P and P′ are parallel to DCD′, and those at D and D′ are parallel to PCP′.

∴ these four tangents form a parallelogram.

The area of parallelogram $=4$ area of parallelogram DP

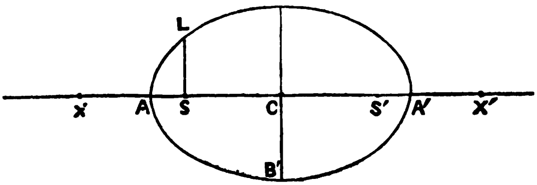
$$=4\text{PF}.\text{CD}$$
$$=4.\text{BC}.\text{AC}$$
$$=\text{AA}'.\text{BB}'$$

PROP. 18.—*THEOREM.*

82. *If* ACA′ *be the axis,* C *the centre of an ellipse, then*

$$\text{CS}:\text{CA}::\text{CA}:\text{CX}::\text{SA}:\text{AX}$$

Fig. 39.



For $SA : AX :: SA' : A'X$

$\therefore CA-CS : CX-CA :: CA+CS : CX+CA :: SA : AX$

$\therefore CS : CA :: CA : CX :: SA : AX$ Q.E.D.

Note $CS . CX = CA^2$

PROP. 19.—*THEOREM.*

83. *If* S *be the focus and* B *the extremity of the conjugate axis,*

$$SB = AC \textit{ and } BC^2 = AS . SA'$$

Join SB in the figure of Art. 82.

$$\begin{aligned} SB : CX &:: SA : AX \\ &:: CA : CX \\ \therefore\ SB &= CA \end{aligned}$$

Hence $BC^2 = AC^2 - CS^2$
$= AS . SA'$ [Euc. ii. 5. Cor.]

84. If we take two points S′ and X′ on the axis such that $A'S' = AS$ and $A'X' = AX$, it will be clear, from the symmetry of the ellipse, that we have a second focus and directrix.

We also notice, since $CA : CX :: SA : AX$, that the circle on AA′ as diameter is a particular case of the 'Auxiliary Circle.'

Def.—The transverse axis is sometimes called the 'Major Axis,' and the conjugate axis the 'Minor Axis,' for since SB is equal to AC it is clear that BC is less than AC.

85. And since $SL^2 : AS . SA' :: BC^2 : AC^2$ [Art. 18]

$\therefore SL^2 : BC^2 :: BC^2 : AC$

and $SL : BC :: BC : AC$

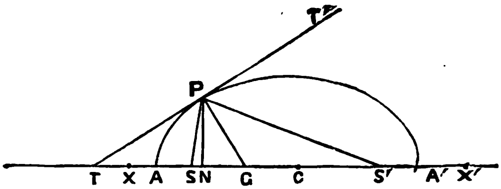
or $SL . AC = BC^2$

PROP. 20.—*THEOREM.*

86. *The sum of the focal distances of any point is equal to the transverse axis.*

Let P be any point. Draw the ordinate PN.

FIG. 40.



Because $\mathrm{SP : NX :: SA : AX :: CA : CX}$

and $\mathrm{S'P : NX' : S'A' : A'X' :: CA : CX}$

$\therefore\ \mathrm{SP + S'P : NX + NX' :: 2CA : 2CX}$

But $\mathrm{NX + NX' = 2CX}$

$\therefore\ \mathrm{SP + S'P = 2CA = AA'}$ Q.E.D.

87. *Cor.*—Hence it appears that SP + S′P + SS′ is constant, and therefore that if an endless string of this length be placed over two pegs at S and S′ and kept tight by a pencil at P, then the pencil will trace out the ellipse, which gives another mechanical construction for the curve.

88. From this it is also evident that the sum of the distances of a point from the foci is greater or less than AA′ as the point is without or within the ellipse.

PROP. 21.—*THEOREM.*

89. *If* P *be any point on an ellipse whose foci are* S *and* S′, *then* SP, S′P *are equally inclined to the tangent and normal.*

Let the normal at P meet the axis in G

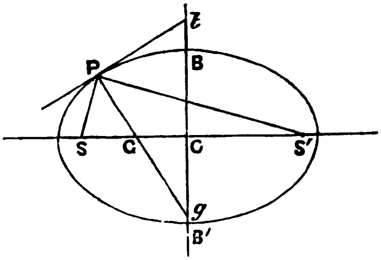
$$\begin{aligned} \mathrm{SG : SP} &:: \mathrm{SA : AX} \\ &:: \mathrm{S'A' : A'X'} \\ &:: \mathrm{S'G : S'P} \\ \therefore\ \mathrm{SG : S'G} &:: \mathrm{SP : S'P} \end{aligned}$$

$\therefore$ SP, S′P are equally inclined to PG [Euc. vi. 3]

and $\therefore$ also equally inclined to the tangent at P Q.E.D.

90. *Cor.*—Let the tangent and normal at P cut the conjugate axis in t and g respectively.

FIG. 41.



Since ∠ SPG = ∠ S′PG, and that equal angles in a circle stand on equal arcs, if a circle be drawn round S, P, S′ the arc which stands on SS′ is bisected by the normal PG, as also by CB′ or CB′ produced. Hence g lies on this circle.

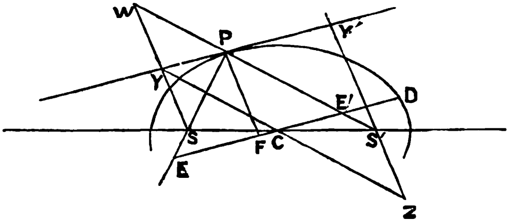
And since the centre of the circle must lie on BB′ and that gPt is a right angle, t also lies on the circle.

PROP. 22.—*THEOREM.*

91. *The perpendiculars from the foci on any tangent meet the tangent on the auxiliary circle of the centre, and the semi-axis minor is a mean proportional between their lengths.*

Let SY, S′Y′ be the perpendiculars to the tangent at P. Join SP, S′P and let S′P, SY meet in W. Join CY.

FIG. 42.



Because $\angle SPY = \angle WPY$ [Art. 89]

$$\therefore\ SY = WY \text{ and } SP = WP$$

Again $S'W = S'P + PW = S'P + SP = 2CA$

$\therefore$ CY, which joins the middle points of SW and SS′, must be equal to CA. Hence Y lies on the auxiliary circle of the point C. In the same way Y′ may be shown to lie on the auxiliary circle of the point C.

Produce YC to meet Y′S′, produced in Z′.

Because YY′Z′ is a right angle and YC passes through the centre of the circle $\therefore$ Z′ lies on the circle

and $S'Z' = SY$

$$\begin{aligned}\therefore\ SY \,.\, S'Y' &= S'Z' \,.\, S'Y' \\ &= A'S' \,.\, AS' \\ &= BC^2 \quad \text{[Art. 83.]}\end{aligned}$$

Q.E.D.

92. *Cor.*—If the diameter conjugate to CP or parallel to the tangent at P meet SP, S′P in E, E′, then

$$PE' = CY = CA$$

and similarly $PE = CY' = CA$

PROP. 23.—*THEOREM.*

93. *If* SP, S′P *be the focal distances of any point* P, *and* CD *be the semi-diameter conjugate to* CP, *then*

$$SP \,.\, S'P = CD^2$$

For in figure 52, Art. 91,

the triangles SPY, S′PY′, PEF are similar

$$\therefore\ SP : SY :: PE : PF :: AC : PF$$

and $S'P' : S'Y' :: PE : PF :: AC : PF$

$$\therefore\ SP \,.\, S'P : BC^2 :: AC^2 : PF^2$$

But $PF \,.\, CD = AC \,.\, BC$ [Art. 79]

$$\therefore\ CD^2 : BC^2 :: AC^2 : PF^2$$

$$\therefore\ SP \,.\, S'P = CD^2$$

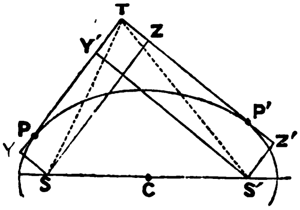
Q.E.D.

PROP. 24.—*THEOREM.*

94. *If* TP, TP′ *be a pair of tangents drawn from* T, *then* STP=S′TP′ *and* S′TP=STP′

From S and S′ draw perpendiculars to the tangents meeting them in Y, Y′ and Z, Z′ respectively. Join ST, S′T.

Fig. 43.



$$SY \,.\, S'Y' = BC^2 = SZ \,.\, S'Z'$$
$$\therefore\ SY : SZ :: S'Z' : S'Y'$$

Hence ST and S′T divide the angle PTP′ in the same ratio.
Q.E.D.

$$\therefore\ STP = S'TP' \text{ and } STP' = S'TP$$

PROP. 25.—*THEOREM.*

95. *The locus of intersection of tangents at right angles is a fixed circle.*

If the angle PTP′ be a right angle (using figure 43, article 94), TY is parallel to SZ and TZ to S′Y′. And since the points Y, Y′, Z, Z′ lie on a circle whose radius is CA

$$\begin{aligned}\therefore\ CT^2 &= CA^2 + TY \,.\, TY' \\ &= CA^2 + SZ \,.\, S'Z' \\ &= CA^2 + BC^2 = \text{a constant for all positions of T.}\end{aligned}$$

Hence tangents at right angles always intersect on a fixed circle. Q.E.D.

CHAPTER IV.

THE HYPERBOLA.

Def.—The curve traced out by a point which moves in such a manner that its distance from a given point is in a constant ratio, *greater than unity*, to its distance from a given straight line is called an 'Hyperbola.'

TRACING THE CURVE.

96. In SX take a point A such that the ratio SA : AX is equal to the given ratio, then A is evidently a point on the curve.

FIG. 44.

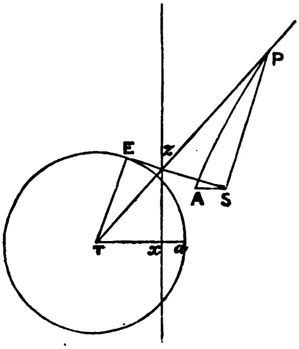
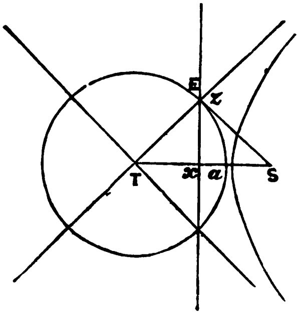
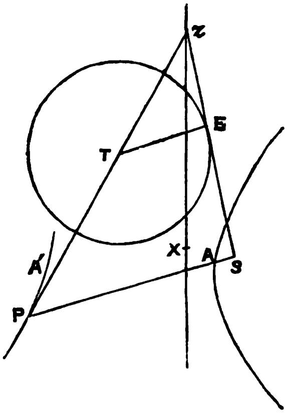


FIG. 45.



Draw the auxiliary circle at any point T on the side of the directrix remote from the focus; let SE, TP be corresponding tangents to the circle and hyperbola respectively. Since the eccentricity is greater than unity T*a* is greater than T*x* $\therefore$ the circle will cut the directrix and be divided into two segments.

FIG. 46.



Three cases will then occur :—

1. When E is on the arc more remote from S the point P will be on the same side of the directrix as the focus. As the auxiliary circle is made to move upwards the point E approaches Z and the angle between SP and ZP slowly diminishes, and $\therefore$ the point of intersection P recedes from the focus.
2. As the circle continues to move upwards E becomes coincident with Z on the directrix, and the point P is then at an infinite distance.
3. As the circle continues to move still further upwards, the point E crosses the directrix and comes between Z and S on the arc of the auxiliary circle nearer the focus; the angle between SP and ZP begins to increase, and the point P again approaches the focus, but now on the opposite side of the directrix, and finally cuts the axis at a point A′ such that SA′ : A′X :: SA : AX.

Hence we come to the following conclusion as to the shape of the curve :—

There are two infinite branches which lie on opposite sides of the directrix, and these branches lie between two intersecting lines which *touch* them at infinity and intersect the axis in a point T such that the angle SZT is a right angle and

$$TZ : TX :: SA : AX$$

These lines are called 'Asymptotes' to the curve; and the construction shows that there are only two such lines.

The curve is also divided symmetrically by the axis.

DEFINITIONS.

1. The line AA′ is called the transverse axis of the hyperbola.

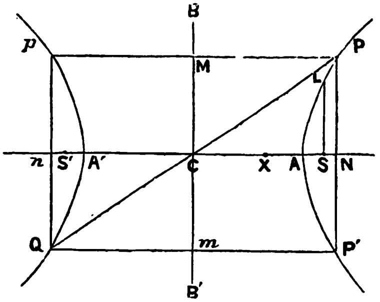
2. The middle point of the transverse axis is called the centre.

PROP. 1.

97. *The symmetry of the Hyperbola.*

Draw any ordinate PN to the transverse axis.

FIG. 47.



A line through C perpendicular to the axis does not meet the curve, but if we mark off CB=CB′ such that the ratio

$CB^2 : CA^2$ represents the constant ratio $PN^2 : AN . A'N$, then the line BB′ may be conveniently called the ‘conjugate axis.’ Exactly as in Art. 49 we deduce the symmetry of the hyperbola.

98. *Cor.* 1.—$PN^2 : AN . A'N :: BC^2 : AC^2$

$$\therefore\ PN^2 : CN^2 - CA^2 :: BC^2 : AC^2$$
$$\therefore\ CM^2 : PM^2 - CA^2 :: BC^2 : AC^2$$
$$\therefore\ CM^2 : BC^2 :: PM^2 - CA^2 : AC^2$$
$$\therefore\ CM^2 + BC^2 : BC^2 :: PM^2 : AC^2$$

or
$$PM^2 : CM^2 + BC^2 :: AC^2 : BC^2$$

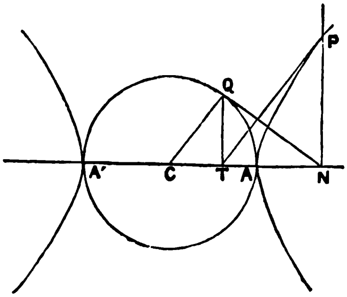
Cor. 2.—From the symmetry of the hyperbola we also infer that the two asymptotic lines pass through the centre.

PROP. 2.

99. *If* PN *be an ordinate of an hyperbola and* NQ *be a tangent to the circle described on the transverse axis as diameter, then*

$$PN : QN :: BC : AC$$

FIG. 48.



For
$$PN^2 : AN . A'N :: BC^2 : AC^2$$
and
$$AN . A'N = QN^2$$
$$\therefore\ PN^2 : QN^2 :: BC^2 : AC^2 \qquad \text{Q.E.D.}$$

100. *Cor.*—If PN be produced to p so that $pN=QN$, then $pN^2 : AN . A'N :: AC^2 : AC^2$

and p lies on an hyperbola having equal axes.

Def.—An hyperbola whose axes are equal is called an 'Equilateral Hyperbola.'

Cor.—As in Art. 54, Cors. 1 and 2, it may be shown that the tangents at p and P meet the axis in the same point T.

PROP. 3.

101. *If the tangent at* P *meet the axis in* T, *and* PN *be the ordinate, it may be proved in exactly the same way as in Art.* 60, *using a figure similar to that in Art.* 110, *that*

$$CT . CN = CA^2$$

102. This points to the conclusion that QT [Art. 99] is perpendicular to the axis, because

$$CT . CN = CA^2 = CQ^2$$

103. It may also be useful to remark here that since an equilateral hyberbola is one whose axes are equal, and a circle is an ellipse whose axes are equal, the Cor. to Art. 100 corresponds to the Cor. of Art. 54.

PROP. 4.

104. *If the normal at* P *meet the axis in* G, *and* PN *be the ordinate,*

$$NG : CN :: BC^2 : AC^2$$

For, using the figure and construction of Art. 99,

$$PN^2 = TN . NG \qquad QN^2 = TN . NC$$

But $PN^2 : QN^2 :: BC^2 : AC^2$

Hence $NG : CN :: BC^2 : AC^2$

105. The next three propositions are so nearly like the corresponding ones in the ellipse that it will only be necessary to enunciate them.

PROP. 5.

106. *If* PCP′, QCQ′ *be any two diameters, then* PP′ *shall be parallel to* QQ′.

Cor.—The tangents at the extremities of any diameter are parallel.

PROP. 6.

A diameter bisects all chords parallel to the tangents at its extremities.

PROP. 7.

107. *If* QVQ′ *be a double ordinate to the diameter* PCP′, *then* $QV^2 : PV \,.\, P'V$ *in a constant ratio* [figure 49, Art. 110].

108. A line through C parallel to QQ′ does not meet the curve, but if we mark off CD=CD′ such that the ratio $CD^2 : CP^2$ represents the constant ratio $QV^2 : PV \,.\, P'V$, then the line DD′ may be conveniently called the diameter parallel to QQ′ [fig. 49, Art. 110].

PROP. 8.

109. *If* V*p* *be the tangent to the circle on the diameter* PP′ *and* VQ *be the ordinate, then, as in Prop.* 2,

$$QV : pV :: CD : CP$$

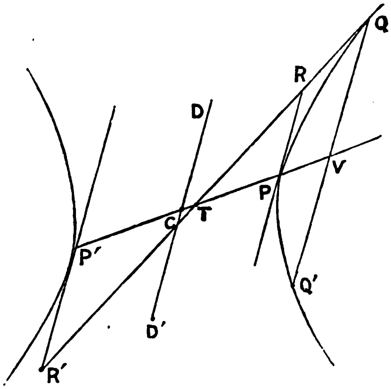
PROP. 9.

110. *If the tangent at* Q *meet any diameter* PP′ *in* T, *and* QV *be the ordinate to that diameter,*

$$CV \,.\, CT = CP^2$$

Let the tangent at Q meet those at P and P′ in the points R and R′ respectively; then

Fig. 49.



Exactly as in the case of the ellipse

$$RP : R'P' :: RQ : R'Q$$

Hence $$TP : TP' :: PV : VP'$$

or $$CP-CT : CP+CT :: CV-CP : CV+CP$$

$$\therefore\ CT : CP :: CP : CV$$

$$\therefore\ CV \,.\, CT = CP^2$$ Q.E.D.

111. *Cor.* 1.—If a circle be described on PP′ as diameter and Vp be the tangent to it, it may be shown, exactly as in Art. 102, that pT is perpendicular to PP′.

Hence $$CV \,.\, VT = Vp^2 = PV \,.\, VP'$$

112. *Cor.* 2.—If the tangent at Q meet the diameter parallel to QV in t, and Qv be drawn parallel to CP meeting that diameter in v, then

$$Cv \,.\, Ct = CD^2$$

For $$Ct : QV :: CT : TV$$

$$\therefore\ Cv \,.\, Ct : QV^2 :: CV \,.\, CT : CV \,.\, TV$$

$$\therefore\ Cv \,.\, Ct : CV \,.\, CT :: QV^2 : PV \,.\, P'V$$

$$:: CD^2 : CP^2$$

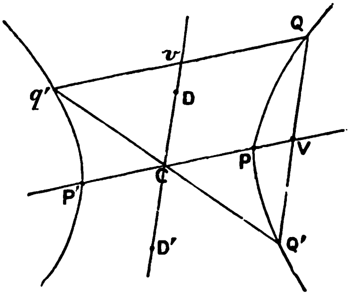
Hence $$Cv \,.\, Ct = CD^2$$

PROP. 10.

113. *If a diameter* PCP′ *bisect chords parallel to another diameter* DCD′, *then shall* DCD′ *bisect all chords parallel to* PCP′.

Draw QQ′ parallel to CD meeting CP in V; draw the diameter Q′C*q*′. Join Q*q*′ meeting CD in *v*.

FIG. 50.



Then, because QV=Q′V and CQ′=C*q*′

∴ Q*q*′ is parallel to CP

Again, since CD is parallel to QQ′ and *q*′C = CQ′

∴ *q*′*v*=Q*v*

∴ DCD′ bisect chords parallel to PCP′. Q.E.D.

Def.—A diameter which bisects all chords parallel to another is said to be 'Conjugate' to it.

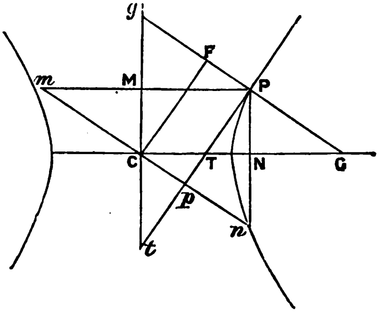
A pair of chords QQ′, Q*q*′ drawn from any point Q to the extremities of a diameter are called 'Supplemental Chords,' and we see that they are parallel to a pair of conjugate diameters.

PROP. 11.

114. *If the normal at a point* P *meet the axes in* G, *g*, *and the diameter conjugate to* CP *in* P, *then shall*

$$PF \cdot PG = BC^2 \text{ and } PF \cdot Pg = AC^2$$

FIG. 51.



Let the tangent at P meet the axes in T, *t*; draw C*p* perpendicular to the tangent meeting the ordinates PM and PN in *m*, *n*.

Because $Cm = PG \quad Cn = Pg$ and $Cp = PF$

$\therefore \; PF \cdot PG = Cp \cdot Cm$
$= Ct \cdot CM$ [Euc. III. 36, cor.]
$= BC^2$ [Art. 112, cor.]

Again $PF \cdot Pg = Cp \cdot Cn$
$= CT \cdot CN$ [Euc. III. 36, cor.]
$= CA^2$ [Art. 101.]

PROP. 12.

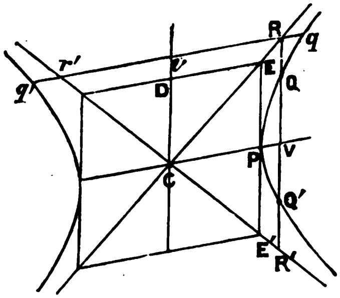
115. *The diagonals of the parallelogram formed by any pair of conjugate diameters* CP, CD *coincide with the asym-*

ptotes; and if any ordinate QVQ′ *of the diameter* CP *meet the asymptotes in* R *and* R′, *then*

$$RV^2 \sim QV^2 = RQ \,.\, RQ' = CD^2$$

and $$RQ = R'Q'$$

FIG. 52.



For $$RV^2 : CV^2 :: CD^2 : CP^2$$

and $$QV^2 : CV^2 - CP^2 :: CD^2 : CP^2$$

$$\therefore\ RV^2 - QV^2 : CP^2 :: CD^2 : CP^2$$

Hence $$RV^2 - QV^2 = RQ \,.\, RQ' = CD^2$$

Again, since CP evidently bisects both QQ′ and RR′

$$\therefore\ RQ = R'Q' \text{ and } RQ' = R'Q$$

Lastly, CE is an asymptote; for since the rectangle RQ . RQ′ is the same for all parallel chords, and RQ′ increases as QQ′ is taken farther and farther from the centre, therefore RQ diminishes. Hence the curve approaches nearer and nearer to the line CE and *touches* it at an infinite distance; therefore CE is an asymptote, and in like manner CE′ is the other asymptote.

Cor. 1.—In a similar manner

$$Rq \,.\, Rq' = CP^2$$

and $$Rq = r'q'$$

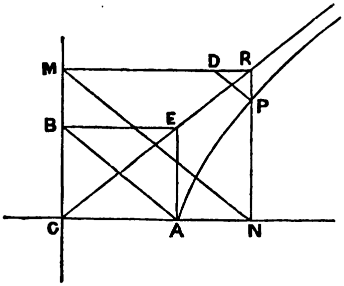
Cor. 2.—If CP and CD be a pair of conjugate diameters, then PD is parallel to one asymptote and is bisected by the other.

PROP. 13.

116. *The difference of the squares on any pair of conjugate diameters is constant.*

From any point R in an asymptote draw the ordinates RPN and RDM to the diameters CA and CB, making PD parallel to AB. Let CE be the diagonal of the parallelogram formed by the conjugate diameters CA and CB.

FIG. 53.



Then $$\begin{aligned} CA : CN &:: CE : CR \\ &:: CB : CM \end{aligned}$$

Hence MN is parallel to AB, and therefore also to PD, and each of these lines is evidently bisected by the asymptote CE. Therefore CP and CD are conjugate diameters.

Again $$RM^2 : RN^2 :: CA^2 : CB^2$$

and $$DM^2 : PN^2 :: CA^2 : CB^2$$

$$\therefore\ RM^2 - DM^2 : RN^2 - PN^2 :: CA^2 : CB^2$$

But $$RN^2 - PN^2 = CB^2 \qquad \text{[Art. 115]}$$

$$\therefore\ RM^2 - DM^2 = CA^2$$

Hence $$(CN^2 + PN^2) - (DM^2 + CM^2) = CA^2 - CB^2$$

$$\therefore\ CP^2 - CD^2 = CA^2 - CB^2 \qquad \text{Q.E.D.}$$

Cor. $$DM^2 = CN^2 - CA^2 = AN \,.\, A'N$$

$$PN^2 = CM^2 - CB^2 = BM \,.\, B'M$$

Also $$PN : DM :: CM : CN :: BC : AC$$

PROP. 14.

117. *The locus of the extremity* D *of diameters which do not meet the curve is an hyperbola.*

For, using the figure of Prop. 13,

$$DM^2 : PN^2 :: CA^2 : CB^2$$
$$\therefore\ DM^2 : BM \,.\, B'M :: CA^2 : CB^2$$

Hence D lies on an hyperbola whose axes are CB and CA.

118. *Def.*—The hyperbola which is the locus of the extremities of diameters of an hyperbola, which do not meet the curve, is called the Conjugate Hyperbola.

PROP. 15.

119. *Conjugate diameters of an hyperbola are also conjugate diameters of the conjugate hyperbola.*

For in figure 62, Prop. 12, let qq' meet the conjugate hyperbola in D' and D''. Then

$$Rq = r'q' \text{ and } RD' = r'D''$$
$$\therefore\ D'q = D''q'$$

But CD bisects qq', which is parallel to CP.

Hence CD also bisects $D'D''$, and therefore CD and CP are conjugate diameters of the conjugate hyperbola. Q.E.D.

Cor.—The conjugate hyperbola has the same asymptotes as the given hyperbola.

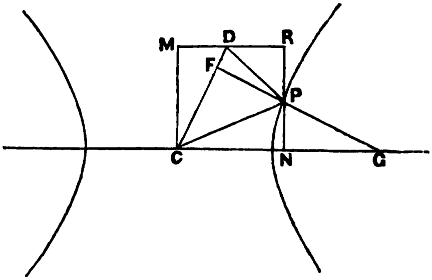
PROP. 16.—*THEOREM.*

120. *If* CP, CD *be conjugate diameters and the normal at* P *meet* CD *in* F,

$$PF \,.\, CD = AC \,.\, BC$$

Draw the ordinates PN and DM, and let PF meet the axis in G.

FIG. 54.



Because PG, PN, NG are respectively perpendicular to CD, DM, CM

∴ PGN and CDM are similar triangles

∴ PG : CD :: PN : DM

:: BC : AC [Art. 116, cor.]

∴ PF . PG : PF . CD :: BC^2 : BC . AC

But PF . PG = BC^2

∴ PF . CD = BC . AC Q.E.D.

121. *Cor.* 1.—The parallelogram formed by drawing tangents to the hyperbola and its conjugate at the extremities of a pair of conjugate diameters is of constant area.

For in figure 62, Art. 115, the area of the parallelogram formed by drawing tangents at the extremities of the diameters PCP′ and DCD′

= 4 . PCDE

= 4 . PF . CD

= 4 . AC . BC

= AA′ . BB′

122. *Cor.* 2.—Again, the area of the triangle ECE′

= AC . BC

or the triangle which any tangent to an hyperbola forms with the asymptotes is of constant area.

PROP. 17.—*THEOREM.*

123. *If* ACA′ *be the transverse axis,* C *the centre,* S *the focus, and* X *the foot of the directrix,*

$$CS : CA :: CA : CX :: SA : AX$$

Using the figure of Art. 97, we get

$$SA : AX :: SA' : A'X$$

$$\therefore\ CS-CA : CA-CX :: CS+CA : CA+CX$$

$$\therefore\ CS : CA :: CA : CX :: CS-CA : CA-CX$$

$$:: SA : AX \qquad \text{Q.E.D.}$$

PROP. 18.—*THEOREM.*

124. *If* S *be the focus and* B *the end of conjugate axis,*

$$BC^2=AS \, . \, A'S, \textit{ and } CS^2=CA^2+CB^2$$

Draw the ordinate SL through the focus [fig. 47, Art. 97].

$$SL : SA :: SX : AX :: SA+AX : AX$$

$$:: CS+CA : CA$$

Similarly $SL : SA' :: SX : A'X :: SA'-A'X : A'X$

$$:: CS-CA : CA$$

$$\therefore\ SL^2 : SA \, . \, SA' :: CS^2-CA^2 : CA^2$$

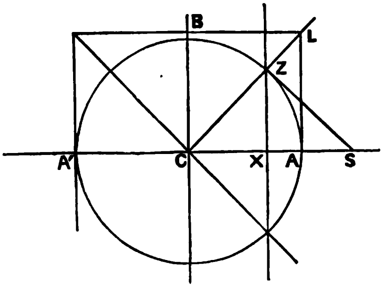
But $SL^2 : SA \, . \, SA' :: BC^2 : AC^2$

$$\therefore\ BC^2=CS^2-CA^2$$

$$=SA \, . \, SA' \qquad \text{Q.E.D.}$$

125. Since CA : CX :: SA : AX the circle on AA′ as diameter is the auxiliary circle of the point C.

Fig. 55.



And since $CS \cdot CX = AC^2 = CZ^2$

$\therefore$ CZS is a right angle

$\therefore$ SZ is a tangent to the circle

and CZ is one of the asymptotes.

Cor. 1. $SZ^2 = SA \cdot SA'$

$= BC^2$

$\therefore SZ = BC$

Cor. 2.—If the tangent at A meet the asymptote in L

$AL = SZ = BC$ and $CL = CS$

Hence also the asymptotes are the diagonals of the rectangle formed by the axes; which was also proved in Art. 115.

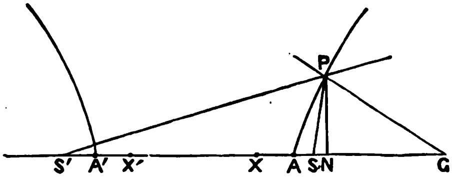
126. It is evident from the symmetry of the hyperbola that if we take two points S′ and X′ such that $S'A' = SA$ and $A'X' = AX$, there will be another focus and directrix. Also since $CA : CX :: SA : AX$ the circle on the transverse axis is an auxiliary circle.

PROP. 19.—*THEOREM.*

127. *The difference of the focal distances of any point is equal to the transverse axis of the hyperbola.*

Let P be any point; draw the ordinate PN.

FIG. 56.



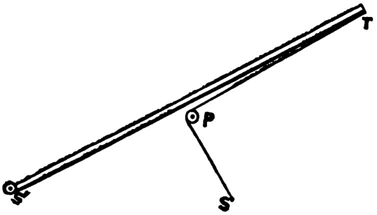
$$S'P : NX' :: SA : AX :: CA : CX$$
$$SP : NX :: SA : AX :: CA : CX$$
$$S'P - SP : NX' - NX :: 2CA : 2CX$$

But $NX' - NX = 2CX$

$\therefore S'P - SP = 2CA$ Q.E.D.

128. Also as in an ellipse we can deduce a mechanical construction for the curve.

FIG. 57.



Let S′T be a rod moveable in the plane of the paper about a fixed point S′. Let a flexible string TPS be tied to the other end of the rod and to a fixed point S, the length of the rod exceeding the length of the string by any given quantity. Let this string be held tight against the rod by a pencil P, and the system turned about S′. Then P will trace out an hyperbola with foci S′ and S; for

$$\begin{aligned} S'P - SP &= (S'P + PT) - (SP + PT) \\ &= \text{length of rod} - \text{length of string} \\ &= \text{constant} \end{aligned}$$

PROP. 20.

129. *If* P *be any point on an hyperbola whose foci are* S *and* S′, *then* SP, S′P *are equally inclined to the tangent and normal.*

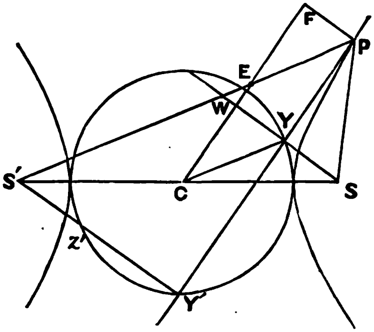
Using figure of [Art. 127] the proof is exactly the same as [Art. 89].

PROP. 21.

130. *The perpendiculars from the foci on any tangent meet the tangent on the auxiliary circle of the centre, and the semi-axis minor is a mean proportional to their lengths.*

Let SY, S′Y′ be the perpendiculars to the tangent at P. Join SP, S′P and let S′P and SY meet in W. Join CY.

FIG. 58.



The proof is the same as [Art. 91] with the exception of $S'W = S'P - PW = S'P - SP = 2CA$.

PROP. 22.

131. *If* SP, S′P *be the focal distances of any point* P, *and* CD *the semidiameter conjugate to* CP; *then*

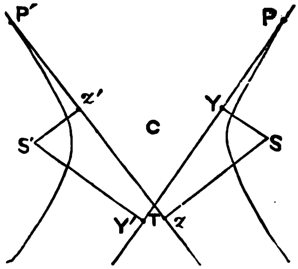
$$SP \,.\, S'P = CD^2$$

Using the figure of [Art. 130] the proof is the same as [Art. 93].

PROP. 23.

132. *If two tangents be drawn from any point* T *to an hyperbola they are equally inclined to* ST *and* S′T.

FIG. 59.



The construction and proof are exactly the same as Art. 94.

133. *Cor.*—If TP and TP′ be at right angles, then

$$SY=TZ \text{ and } S'Y'=TZ'$$

Again, since the points Y, Y′, Z, Z′ all lie on the circle on AA′,

$$\begin{aligned} CT^2 &= CA^2-TZ\,.\,TZ' \\ &= CA^2-SY\,.\,S'Y' \\ &= CA^2-CB^2 \end{aligned}$$

Hence tangents at right angles intersect on a fixed circle.

CHAPTER V.

THE RECTANGULAR HYPERBOLA.

134. If the axes of an hyperbola be equal, the angle between the asymptotes is a right angle, and the hyperbola is called 'equilateral' or 'rectangular.'

Many of the properties of the hyperbola will become modified in the case of the rectangular hyperbola, and many new ones will suggest themselves.

The property in [Art. 97] becomes, since $BC=AC$,

$$\begin{aligned} PN^2 &= AN \cdot A'N \\ &= CN^2 - CA^2 \end{aligned}$$

135. Hence if the ordinate PN of an hyperbola meet a rectangular hyperbola having the same transverse axis in the point Q,

$$PN^2 : QN^2 :: BC^2 : AC^2$$

a property corresponding to the one in Art. 53.

136. From Art. 124,

$$CS^2 = 2AC^2$$

$$\therefore\ CS : CA :: \sqrt{2} :: SA : AX$$

which determines the eccentricity of a rectangular hyperbola to be $\sqrt{2}$,

137. From Art. 115,

Since RCR′ is a right angle, P and V are the centres of the circles round the triangles ECE′ and RCR′.

Hence $\qquad CP=PE=CD$

that is, conjugate diameters are equal.

Again, if a diameter be drawn at right angles to CD, it will be equally inclined to the axis with CP and ∴ equal to it; hence also diameters at right angles are equal to one another.

Again, $RV = CV$

But $RV^2 - QV^2 = CD^2$

$\therefore\ CV^2 - QV^2 = CP^2$

Again, $\angle VCR = \angle VRC$

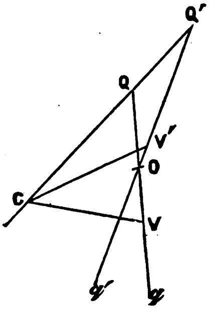
Hence a chord QQ′ and its conjugate CV are equally inclined to the asymptote.

PROP. 1.

138. *The angle between any two chords is equal to that between their conjugates.*

For if CQQ′ be the asymptote, V V′ the middle points of the chords,

FIG. 60.

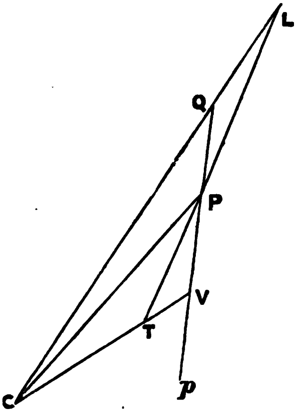


$$\angle VCQ = \angle VQC$$
$$\angle V'CQ' = \angle V'Q'C$$
$$\therefore\ \angle VCV' = \angle QOQ' \qquad \text{Q.E.D.}$$

139. *Cor.*—A particular case of the last article is when one of the chords becomes a tangent.

Let P*p* be the chord meeting the asymptote in Q. Let CV be its conjugate. Let the tangent at P meet the asymptote in L and CV in T.

FIG. 61.



$$\angle VCQ = \angle VQC$$
$$\angle PCQ = \angle PLC$$
$$\therefore \angle PCV = \angle VPT$$

140. From Art. 114

$$PF \cdot PG = PF \cdot Pg$$
$$\therefore PG = Pg$$

From Art. 120

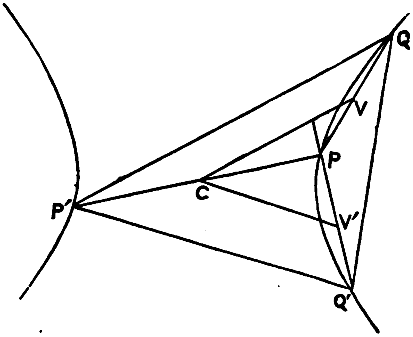
$$PF \cdot CD = AC^2 = PF \cdot PG$$
$$\therefore PG = Pg = CD$$

PROP. 2.

141. *A chord subtends at the extremities of any diameter angles which are either equal or supplementary.*

Let QQ′ be the chord, PCP′ the diameter; bisect PQ, PQ′ in V and V′. Join CV, CV′.

FIG. 62.



Then CV, CV′ are respectively parallel to P′Q, P′Q′.

$$\therefore \angle QP'Q' = \angle VCV'$$

But VCV′ is equal to the angle between PQ and Q′P produced [Art. 138].

Hence QPQ′ is the supplement of QP′Q′.

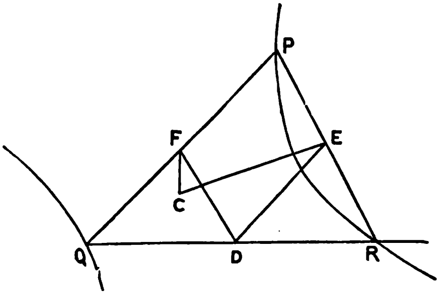
In the same way it may be proved when Q and Q′ are on different branches of the curve.

PROP. 3.

142. *If a rectangular hyperbola pass through the angular points of a triangle its centre will lie on the nine point circle of the triangle.*

Let PQR be the triangle; D, E, F, the middle points of its sides; C the centre of the hyperbola. Join CE, CF, DE, and DF.

FIG. 63.



Then because CE, CF are conjugate to PR, QP;

$$\therefore \angle FCE = \angle QPR \qquad \text{[Art. 138.]}$$
$$= \angle EDF$$

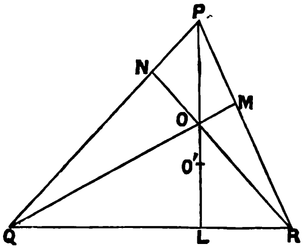
∴ The circle passing through EDF, also passes through C.

∴ C lies on the nine point circle of the triangle.

143. Also *this rectangular hyperbola passes through the orthocentre of the triangle.*

For let O be the orthocentre, and LMN the feet of perpendiculars.

FIG. 64.



Let the curve cut PL in O′.

Then $LQ . LR = LP . LO'$ [Art. 18 and 137.]

But QOL, PRL are similar triangles,

whence $LQ : LO :: LP : LR$

and $\therefore LQ . LR = LP . LO$

Wherefore O and O′ are coincident, and the curve passes through the orthocentre O. Q.E.D.

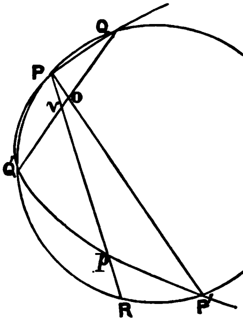
CHAPTER VI.

CURVATURE OF CONICS.

EXPLANATION OF CURVATURE.

144. If two chords of a conic, PP′, QQ′, be taken equally inclined to the axis and intersecting in O, the chords parallel to them through the focus will be equal and divided into equal parts.

FIG. 65.



$\therefore$ OP . OP′=OQ . OQ′ [Art. 18.]

Hence a circle can be described through the four points Q, P, Q′, P′.

145. If two of these points, P, Q′, be taken very near to one another, PQ′ produced will become a common tangent to the circle and conic, and the two are said to have contact of the first order, or to touch one another.

146. If the *three* points Q, P, Q′ be taken very near to one another, PQ′ and PQ will both become tangents, and the curves have *two* consecutive tangents common, and are said to have contact of the *second* order, or to have the same 'curvature.'

Def.—The circle which has the same curvature as a curve at any point is called the circle of curvature, and any chord of the circle drawn through that point is called a chord of curvature.

The radius of the circle of curvature is called the radius of curvature.

PROP. 1.—*PROBLEM.*

147. *To find an expression for the chord of curvature in terms of the chord of the conic.*

Let a circle intersect the conic in the four points Q, P, Q′, P′. Through P draw a chord cutting QQ′, the conic, and the circle respectively in V, p, R. Let t, c stand for the squares of the semidiameters parallel to QQ′ and the chord of the circle respectively, or for the parallel focal chords.

Then $$\mathrm{QV}\,.\,\mathrm{Q'V} : \mathrm{PV}\,.\,p\mathrm{V} :: t : c \qquad \text{[Art. 27]}$$
$$\therefore\ \mathrm{PV}\,.\,\mathrm{VR} : \mathrm{PV}\,.\,p\mathrm{V} :: t : c$$
$$\therefore\ \mathrm{VR} : p\mathrm{V} :: t : c$$

Now if Q, P, Q′ be taken very near to one another, QQ′ becomes a tangent and PR the chord of curvature, and V coincides with P, and we get

$$\mathrm{PR} : \mathrm{P}p :: t : c$$

PROP. 2.—*THEOREM.*

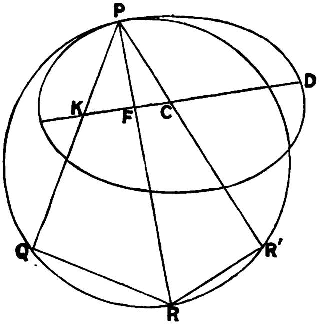
148. *If* PQ *be the chord of curvature in any direction, of an ellipse or hyperbola, meeting* CD, *the diameter conjugate to* CP *in* K, *then*

$$\mathrm{PQ}\,.\,\mathrm{PK} = 2\mathrm{CD}^2$$

Let PR′ be the chord of curvature through the centre; let

PR be the diameter of curvature meeting CD in F. Join RR′, RQ.

FIG. 66.



$$PR' : 2CP :: CD^2 : CP^2 \qquad \text{[Art. 147.]}$$

$$\therefore\ PR' . CP = 2CD^2$$

Since circles will go round the quadrilaterals FCR′R and FKQR,

$$\therefore\ PQ . PK = PF . PR = PC . PR' = 2CD^2 \quad \text{Q.E.D.}$$

PROP. 3.—*THEOREM.*

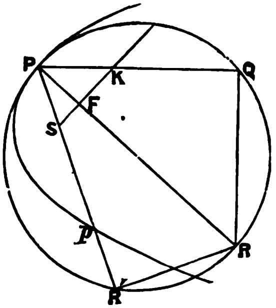
149. *If* PQ *be the chord of curvature in any direction, of a parabola, meeting the focal chord, parallel to the tangent at* P, *in* K,

$$PQ . PK = 4SP^2$$

And the chord of curvature through the focus$=4SP$.

Let PR′ be the chord of curvature through the focus. Let PR be the diameter of curvature meeting the focal chord, parallel to the tangent at P, in F. Join RQ, RR′.

FIG. 67.



$$PR' : Pp :: 4SP : Pp \qquad [\text{Art. } 147.]$$
$$\therefore\ PR' = 4SP$$

[*i.e.* the chord of curvature through the focus is 4SP.]

Since circles will go round PKQR and FSR'R,

$$\therefore\ PQ.PK = PF.PR = PS.PR' = 4SP^2 \qquad \text{Q.E.D.}$$

150. *Note.*—It is proved above that the length of the chord of curvature through the centre is $\frac{2CD^2}{CP}$. If PQ pass through the focus, $PK = AC$. Hence the chord of curvature through the focus $= \frac{2CD^2}{AC}$.

From these propositions the value of the radius of curvature can at once be inferred.

1. Case of central conic.

$$\text{Radius of curvature} = \frac{1}{2}PR = \frac{CD^2}{PF} \quad \dots\dots\dots\dots (\alpha)$$

and
$$\therefore\ = \frac{CD^3}{PF.CD} = \frac{CD^3}{AC.BC} \dots\dots (\beta)$$
$$= \frac{PG^3.AC^3}{AC.BC^4} \quad \left(\text{since } \frac{PG}{CD} = \frac{BC}{AC}\right)$$
$$= \frac{PG^3.AC^2}{BC^4}$$
$$= \frac{(\text{Normal})^3}{(\text{Semi Latus Rectum})^2} \dots (\gamma)$$

2. **Parabola.**

$$\text{Radius of curvature} = \frac{1}{2}PR = \frac{2SP^2}{PF}$$

$$= \frac{2SP^2}{SY} = \frac{2SY^4}{AS^2 . SY}$$

$$= \frac{2SY^3}{AS^2}$$

$$= \frac{1}{4}\frac{PG^3}{AS^2}$$

$$= \frac{(\text{Normal})^3}{(\text{Semi Latus Rectum})^2} \quad . \; . \; . \; (\gamma)$$

Results (α), (β), (γ), should be remembered.

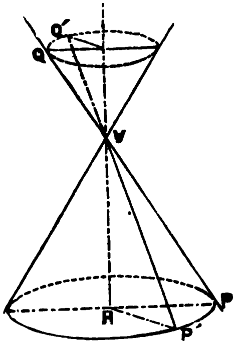
CHAPTER VII.

THE RIGHT CIRCULAR CONE.

Def.—‘A cone is a solid figure described by the revolution of a right-angled triangle about one of the sides containing the right angle, which side remains fixed.’ [Euc. XI. Def. 18.]

151. Let VHP be the right-angled triangle, VH the side which remains fixed. As the triangle revolves about the side VH, the side VP produced infinitely far both ways traces out the surface of a right circular cone.

Fig. 68.



[Dotted lines are those beneath surface of cone or plane of paper.]

Def.—The fixed point V is called the Vertex of the Cone.

The fixed line VH produced infinitely far both ways is called the Axis of the Cone.

Twice the angle PVH is called the Vertical Angle of the Cone.

Any line P′VQ′ drawn through the vertex and on the surface of the cone is called a 'Generator' or 'Generating Line.'

It is clear that if a plane perpendicular to the axis of the cone intersect the surface of the cone, the curve of intersection will be a circle.

Again, the distance of any point, on a circular section, from the vertex is constant, thus

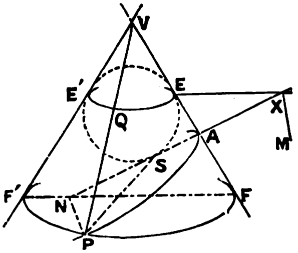
$$VP=VP' \text{ and } VQ=VQ'$$

Also we notice that if a plane passing through the vertex intersect the surface of a cone the curve of intersection will be two generators.

PROP.

152. If a plane which is not perpendicular to the axis and does not pass through the vertex intersect the surface of a cone, the curve of intersection will be an ellipse, parabola, or hyperbola, according to the direction of the plane.

Fig. 69.



Let the plane of the paper contain the axis of the cone and intersect its surface in the generators VF, VF′; let the plane of section be perpendicular to the plane of the paper and intersect it in the line AN, and the surface of the cone in the curve AP. Draw the plane FPF′ perpendicular to the axis of the cone, and also to the plane of the paper; let it intersect the plane of the paper in the line FNF′ and the surface of the cone in the circle FPF′.

Inscribe a sphere in the cone touching the plane of section in the point S and the surface of the cone in the circle EQE′; let the plane of this circle intersect the plane of section in the line XM. Join SP and let VP cut the circle EQE′ in the point Q.

Then, because the plane of section and the plane FPF′ are both perpendicular to the plane of the paper, their intersection PN is perpendicular to the plane of the paper. Similarly XM is perpendicular to the plane of the paper.

Again, SP and PQ are tangents to the sphere.

$$\begin{aligned} \therefore SP &= PQ \\ &= EF \end{aligned}$$

Now from similar triangles we have

$$EF : NX :: AE : AX$$

Hence $$SP : NX :: SA : AX$$

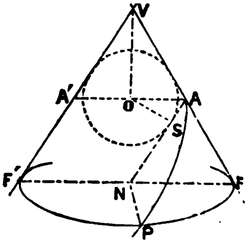
Therefore the curve AP is either an ellipse, a parabola, or an hyperbola, according as SA is less, equal to, or greater than AX.

153. (1) If AN be parallel to the generator VF′, then the angle AXE is equal to the angle VE′X; that is, to VEE′ or AEX.

$$\therefore AE = AX$$
$$\text{or } SA = AX$$

Therefore the curve is a parabola.

Fig. 70.



154. *Cor.*—If O be the centre of the inscribed sphere, since PN is perpendicular to the diameter FF′ of the circle FPF,

$$\therefore \text{PN}^2 = \text{NF}' \,.\, \text{NF} \quad \text{[Euc. VI. 8, cor.]}$$
$$= 2\text{OA} \,.\, \text{NF}$$

But from similar triangles

$$\text{NF} : \text{AN} :: 2\text{AO} : \text{AV}$$
$$:: 2\text{AS} : \text{AO}$$
$$\therefore \text{OA} \,.\, \text{NF} = 2\text{AS} \,.\, \text{AN}$$

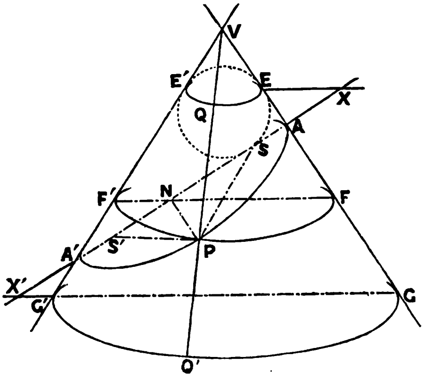
Therefore $\text{PN}^2 = 4\text{AS} \,.\, \text{AN}$

155. (2) If AN meet the generator VF′ in A′ on the same side of the vertex as A, then the angle AXE is less than the angle AEX; therefore

AE or SA is less than AX

therefore the curve is an ellipse.

Fig. 71.



In this case another sphere can be inscribed in the cone [illegible]uching the plane of section in S′, and the cone in the circle GQ′G′; and it can easily be shown that S′ is the other focus of the ellipse, and that the intersection of the plane of section with the plane of the circle GQ′G′ is the other directrix.

Cor. 1.—To prove that SA=S′A′ and that

$$\text{EG} = \text{E}'\text{G}' = \text{AA}'$$

also $$\text{SP} + \text{S}'\text{P} = \text{AA}'$$

For since $$EG=E'G'$$

$$\therefore EA+AG=E'A'+A'G$$

Hence $$AS+AS'=A'S+A'S'$$

or $$2AS+SS'=SS'+2A'S'$$

$$\therefore AS=A'S'$$

Hence $$EG=E'G'=AA'$$

Next let VP meet the circle GQ'G' in Q'. Then since S'P and PQ' are tangents to the same sphere

$$\therefore S'P=PQ'$$

also $$SP=PQ$$

$$\therefore SP+S'P=QQ'=EG=AA'$$

156. *Cor.* 2.—To prove that

$$PN^2 : AN \,.\, A'N :: SX^2 : AX \,.\, A'X$$
$$:: BC^2 : AC^2$$

From similar triangles

$$FN : AN :: EX : AX$$

and $$F'N : A'N :: E'X : A'X$$

$$\therefore FN \,.\, F'N : AN \,.\, A'N :: EX \,.\, E'X : AX \,.\, A'X$$
$$\therefore PN^2 : AN \,.\, A'N :: SX^2 : AX \,.\, A'X$$

and in the same way if C be the middle point of AA' and BC the ordinate,

$$BC^2 : AC^2 :: SX^2 : AX \,.\, A'X$$

157. *Cor.* 3.—If AM and A'M' be the perpendiculars from the vertices A and A' upon the axis of the cone,

$$AM \,.\, A'M'=BC^2$$

For if we imagine PN in the figure to coincide with BC, then clearly FN and F'N are respectively equal to A'M' and AM.

$$\therefore BC^2=FC \,.\, F'C$$
$$=AM \,.\, A'M'$$

158. (3) If AN meet the generator VF' in A' on the opposite side of the vertex to A, then the angle AXE is greater than the angle AEX; and therefore

SA is greater than AX

Therefore the section is an hyperbola.

In this case another sphere can be inscribed in the cone on the other side of the vertex touching the plane of section in S′ and the cone in the circle GQG′, and it may be shown that the plane of section cuts the other portion of the cone in the other branch of the hyperbola, and that S′ is the corresponding focus and the intersection of the plane of section with the plane GQ′G′ is the corresponding directrix.

Cor. 1.—In the same way as in the case of the ellipse

$$SA = S'A'$$

$$EG = E'G' = AA'$$

and

$$S'P - SP = AA'$$

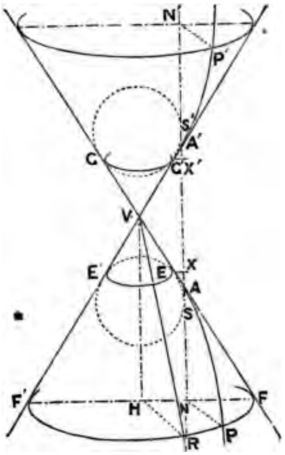
159. *Cor.* 2.—In the same way as in the ellipse

$$PN^2 : AN \,.\, A'N :: SX^2 : AX \,.\, A'X$$
$$:: BC^2 : AC^2$$

160. *Cor.* 3.—If AM, A′M′ be the perpendiculars from the vertices A, A′ upon the axis of the cone,

$$AM \,.\, A'M' = BC^2$$

Fig. 72.



For from similar triangles

$$2AM : AA' :: E'X : A'X$$
$$2A'M' : AA' :: EX : AX$$

Hence $AM . A'M' : AC^2 :: SX^2 : AX . A'X$
$$:: BC^2 : AC^2$$
$$\therefore AM . A'M' = BC^2$$

161. *Cor.* 4.—If a plane through the vertex parallel to the plane of section intersect the cone in the generators VR, VR′, these lines shall be parallel to the asymptotes of the hyperbola. For let this plane intersect the plane of the paper in the line VH; then

$$F'H : VH :: F'N : A'N$$

and $$FH : VH :: FN : AN$$

Hence $RH^2 : VH^2 :: PN^2 : AN . A'N$
$$:: BC^2 : AC^2$$

Hence VR is parallel to an asymptote.

CHAPTER VIII.

TRANSVERSALS.

Def.—If A B C D be four points in a straight line, then the ratio AB . CD : AD . BC is called the '*Anharmonic Ratio*' of the four points. For the sake of brevity this is generally expressed by enclosing the four letters in brackets thus

[ABCD] denotes the anharmonic ratio of the four points A B C D taken in the order A, B, C, D.

Def.—If the four points A B C D be joined to any fifth point O not in the same straight line, the four lines meeting at O are called a '*Pencil*,' and each line a '*Ray*' of that pencil.

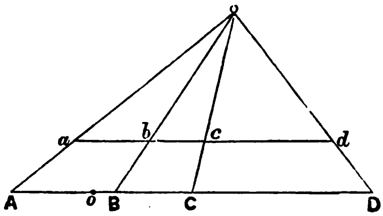
Any line drawn cutting the rays of a pencil is called a *Transversal.*

PROP. 1.

162. *If any transversal cut a pencil in four points the anharmonic ratio of the four points is constant for all positions of the transversal.*

Let OA, OB, OC, OD be the rays of the pencil, and first let the line be parallel to the straight line ABCD and cut these rays in *a*, *b*, *c*, *d*.

FIG. 73.



Then from similar triangles

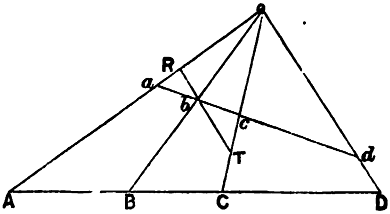
$$AB : AD :: ab : ad$$
$$CD : BC :: cd : bc$$

$$\therefore\ AB.CD : AD.BC : ab.cd : ad.bc$$

or $$[ABCD]=[abcd]$$

Next let *abcd* be not parallel to ABCD. Through *b* draw R*b*T parallel to OD, cutting OA, OC in R and T respectively; then

FIG. 74.



By similar triangles

$$ab : ad :: Rb : Od$$
$$cd : bc :: Od : bT$$
$$ab.cd : ad.bc :: Rb : bT$$

Hence $$[abcd]=Rb : bT$$

and therefore $[abcd]$ is constant for all directions of the straight line through the point *b*; and we have proved that the anharmonic ratio is not altered by moving the transversal parallel to itself, hence it is the same in all directions.

Def.—The anharmonic ratio of a pencil is the same as that of the four points in which any transversal cuts it.

163. The anharmonic ratio of a pencil OA, OB, OC, OD is expressed by O[ABCD].

Def.—In the particular case where the anharmonic ratio of four points is equal to unity, the four points are said to form a *Harmonic Range*; and if the range of four points in which any transversal cuts the rays of a pencil be harmonic, the pencil is called a *Harmonic Pencil.*

PROP. 2.

164. *If four points* A B C D *form a Harmonic Range, then the three straight lines* AB, AC, AD *are in Harmonic Progression.*

For
$$AB . CD = AD . BC$$
$$\therefore AB : AD :: BC : CD$$
or
$$AB : AD :: AC - AB : AD - AC$$
i.e. AB, AC, AD are in harmonic progression.

PROP. 3.

165. *A pencil will be harmonic if a straight line drawn through any point on one of its rays parallel to another, and terminated by the remaining two, is bisected at that point.*

For in the second figure of Art. 162 if
$$Rb = bT$$
$$[abcd] = 1$$

PROP. 4.

166. *The internal and external bisectors of the angle between two straight lines form with them a harmonic pencil.*

For in Art. 162, if OB, OD be the internal and external bisectors of the angle AOC, then BOD is a right angle and RbT is perpendicular to Ob; hence the triangles ObR and ObT are equal; therefore Rb=bT and the pencil is harmonic.

PROP. 5.

167. *A range* ABCD *will be harmonic if it is not altered by interchanging the places of* A *and* C *or of* B *and* D.

For if
$$[ABCD] = [CBAD]$$
then
$$AB . CD : AD . BC :: CB . AD : CD . BA$$
$$:: BC . AD : CD . AB$$
$$\therefore AB^2 . CD^2 : AD^2 . BC^2 :: 1 : 1$$
$$\therefore [ABCD] \text{ is harmonic.}$$

PROP. 6.

168. *If a line* AC *be divided internally and externally in the points* B *and* D *so that* [ABCD] *is harmonic, then if* O *be the middle point of* AC

$$OB\,.\,OD=OC^2$$

For
$$AB\,.\,CD=AD\,.\,BC$$
$$\therefore\ AB : AD :: BC : CD$$
$$\therefore\ OC+OB : OC+OD :: OC-OB : OD-OC$$
$$\therefore\ OB : OC :: OC : OD$$
$$\therefore\ OB\,.\,OD=OC^2$$

PROP. 7.

169. *If two harmonic ranges have one point common, the lines joining the other three pairs of corresponding points all pass through the same point.*

For in fig. 83, Art. 162, if any other transversal $a'bc'd'$ be drawn through b,

then
$$[abcd]=[a'bc'd']$$

and it may easily be proved by *reductio ad absurdum* that the converse of this is true. Thus let aa', cc' intersect in O; join Od' cutting abc in k;

then
$$[a'b'c'd']=[abck]$$
$$\therefore\ [abck]=[abcd]$$
$$\therefore\ ab\,.\,ck : ak\,.\,bc :: ab\,.\,cd : ad\,.\,bc$$
$$\therefore\ ck : ak :: cd : ad$$

That is, ac is divided harmonically in k and d, which is impossible unless k lie between a and c, which it does not.

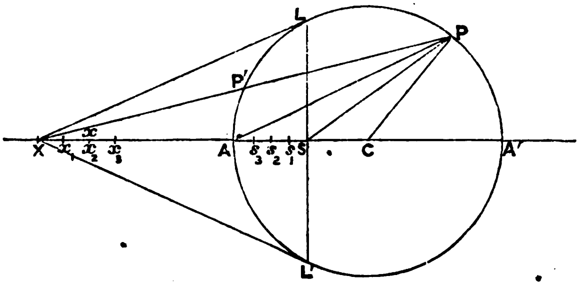
PROP. 8.

170. *If a straight line be drawn from a point cutting a circle, it shall be divided harmonically by the circle and the chord of contact of the point.*

Let X be the point, A'A the diameter which passes through

X, C the centre of the circle, XPP′ the line, and LSL′ the chord of contact cutting AA′ in S.

FIG. 75.



Then XLC is a right angle.

$$CS \,.\, CX = CL^2 = CP^2$$

$$\therefore\ CS : CP :: CP : CX$$

Hence the triangles SCP and PCX are similar.

$$\therefore\ \text{the angle } SPC = \text{the angle } SXP$$

Again $\quad CAP = CPA$

But $\quad CAP = SXP + APX$

and $\quad CPA = SPC + SPA$

Hence $\quad APX = APS$

$$\therefore\ SP : PX :: SA : AX$$

Similarly $\quad SP' : P'X :: SA : AX$

$$\therefore\ SP : PX :: SP' : P'X$$

Hence SX bisects the external angle between P′S and PS produced, and since SL is perpendicular to SX it bisects the angle PSP′.

∴ the pencil S[XP′LP] is harmonic [Art. 166].

As a particular case the range XASA′ is harmonic.

171. *Def.*—If a series of pairs of points X_1, S_1, &c., be taken such that

$$Cs_1 \,.\, Cx_1 = Cs_2 \,.\, Cx_2 = \ldots = CA^2$$

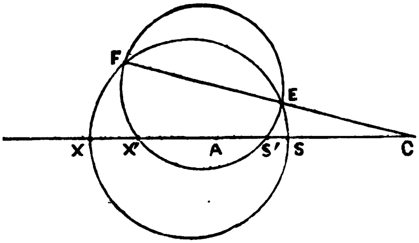
they are said to form a system in Involution. The points A and A′ are called the 'foci,' and C is called the 'centre of Involution.' Any pair of points X and S are said to be Conjugate.

PROP. 9.

172. *Having given two pair of conjugate points, to find the centre and foci of Involution.*

Let S, X and S′, X′ be the two pairs. Through them describe two circles cutting in E and F, and let FE meet the line XS in C. Then

FIG. 76.



$$CS \, . \, CX = CE \, . \, CF$$
$$= CS' \, . \, CX'$$

Hence C is the centre. If CL be the tangent to any one of these circles,

$$CE \, . \, CF = CL^2$$

Hence the foci are found by taking CA and CA′ equal to CL.

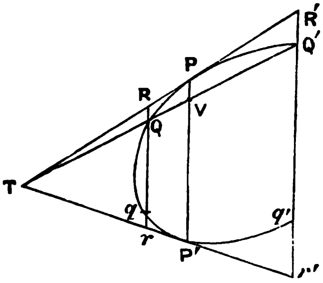
Hence two pairs of points fully determine a system of points in involution.

PROP. 10.

173. *If* TP, TP′ *be a pair of tangents to a conic, and any straight line* TQQ′ *be drawn meeting the curve in* Q *and* Q′ *and the chord of contact in* V, *it shall be harmonically divided.*

Let the chords Qq, $Q'q'$ parallel to PP' meet the tangents in R, r and R', r' respectively.

FIG. 77.



Then $$Qr = qR$$ $$Q'r' = q'R'$$

Again $$QR : Q'R' :: TQ : TQ'$$

also $$Qr : Q'r' :: TQ : TQ'$$

$$\therefore\ QR \,.\, Rq : Q'R' \,.\, R'q' :: TQ^2 : TQ'^2$$

But $$QR \,.\, Rq : Q'R' : R'q' :: RP^2 : R'P^2$$ $$:: QV^2 : Q'V^2$$

Hence $$TQ : TQ' :: VQ : VQ'$$ Q.E.D.

For further explanation of the use of anharmonic ratios the student is referred to Dr. Salmon's works, where the subject is fully treated.

CHAPTER IX.

FOCAL PROJECTION.

174. It has been shown in Chapter I. that if any point P of a conic whose focus is S be joined to a fixed point T, the line PT cutting the directrix of the conic in Z, and if TE be drawn parallel to SP meeting SZ in E, then the point E lies on a circle called the Auxiliary Circle of the point T. [Fig. 5.]

DEFINITIONS.

If S and T be *any* fixed points, and ZX *any* fixed line, the point E is called the 'Focal Projection' of the point P.

The fixed point S is called the 'Focus of Projection,' and the fixed point T the 'Auxiliary Point.'

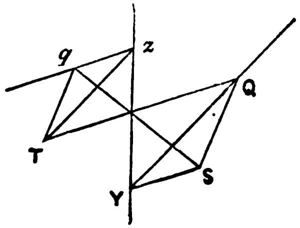
The fixed line ZX is called the 'Directrix of Projection.'

If the focus and directrix of a conic be taken as those of projection, the conic will project into the auxiliary circle of the auxiliary point.

PROP. 1.

175. *The focal projection of a straight line is also a straight line.*

FIG. 78.



Let S be the focus of projection, T the auxiliary point, and zY the directrix of projection. Let QY be the given line meeting the directrix in Y. Draw Tz parallel to QY meeting the directrix in z; through z draw zq parallel to SY, and through T draw Tq parallel to SQ meeting zq in q. Then q shall be the projection of Q.

For from similar triangles SQY and qTz

$$SY : qz :: QY : Tz$$

Hence if Sq and TQ intersect the directrix in K and K′ respectively

$$YK : Kz :: YK' : K'z$$

Therefore Sq and TQ intersect the directrix in the same point, and therefore q is the projection of Q. In the same way any other point on the line QY projects into a point on the line zq which is parallel to SY. Q.E.D.

176. From this proportion we deduce the following:—

1. If any number of lines pass through a point their projections pass through a point—namely, the projection of the given point.
2. If any number of points lie on a straight line, their projections also lie on a straight line.
3. The tangent to any curve projects into a tangent to the projection of the curve, because the line joining two ultimately coincident points projects into a line joining their ultimately coincident projections.
4. The focus projects into the auxiliary point, and any line through the focus into a line through the auxiliary point.
5. Any point on the directrix projects into a point at an infinite distance.
6. Hence also straight lines which intersect in the directrix project into parallel straight lines, and *vice versa.*
7. Lines parallel to the directrix project into lines parallel

to the directrix, because in that case SY becomes parallel to the directrix.

8. Any line QP subtends the same angle at S as its projection qp subtends at T.
9. From [8] it is evident that if ABCD be any four points on a straight line, and $abcd$ their projections,

$$\{ABCD\} = \{abcd\}$$

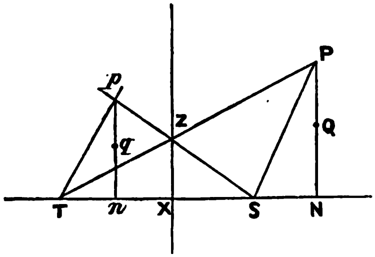
10. A line through S perpendicular to the directrix projects into a line through T also perpendicular to the directrix.

PROP. 2.

177. *Any two segments of a straight line parallel to the directrix of projection project in the same ratio.*

For let pn be the projection of PN, so that SP is parallel to Tp and Sp and TP intersect the directrix in the same point Z.

Fig. 79.



$$\text{Then} \qquad \text{PN} : pn :: \text{SN} : \text{T}n :: \text{SP} : \text{T}p$$
$$:: \text{SZ} : \text{Z}p$$
$$:: \text{SX} : \text{X}n$$
$$:: \text{NX} : \text{TX}$$

all of which ratios are constant for all points on the lines PN and pn.

Hence PQ : pq in a constant ratio Q.E.D.

178. We may notice that if the point X be chosen first, the ratio of PQ to pq may be made equal to any given ratio by properly choosing the point T; or if the point T be first chosen, the ratio of PQ to *pq* may be made anything we please by properly choosing X.

We shall now illustrate this method by a few examples.

179. *Ex.* 1.—Take the focus and directrix of a conic as those of projection, and take any auxiliary point T; then the conic projects into a circle; and since a pair of tangents to the circle subtend equal angles at T, therefore *a pair of tangents to a conic subtend equal angles at* S.

180. *Ex.* 2.—*If* P *be any point on a conic and* ABCD *be four fixed points on it, then shall* P{ABCD} *be constant.*

For project the conic into a circle by taking its focus and directrix as those of projection, and let small letters denote the projections of the capital letters.

FIG. 80.

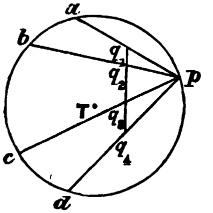
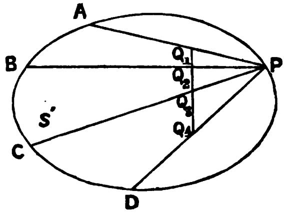


FIG. 81.



Let $Q_1Q_2Q_3Q_4$ and $q_1q_2q_3q_4$ be corresponding transversals.

Then since in the circle the angles which stand on fixed arcs are constant,

$$\therefore \ \{q_1q_2q_3q_4\} \text{ is constant}$$
$$\therefore \text{ also } \{Q_1Q_2Q_3Q_4\} \text{ is constant}$$
$$\therefore \ P\{ABCD\} \text{ is constant}$$

181. *Ex.* 3.—*A conic always projects into a conic.*

For, taking any focus and directrix, if *pabcd* be the projection of PABCD,

$$\{Q_1Q_2Q_3Q_4\} = \{q_1q_2q_3q_4\}$$
$$\therefore \ \{q_1q_2q_3q_4\} = \text{constant}$$
$$\therefore \ p\{q_1q_2q_3q_4\} = \text{constant}$$

Hence the locus of *p* is a conic.

182. *Ex.* 4.—*The angle between two straight lines can be projected into any given angle.*

For if the lines meet the directrix in Y, Y′ the lines project into lines parallel to SY and SY′, and the angle YSY′ can be made any angle we please.

Cor.—In general it is possible to project any two angles simultaneously into given angles.

183. *Ex.* 5.—*Any conic can be projected into a circle, and any line which does not cut it to infinity, simultaneously.*

Fig. 82.

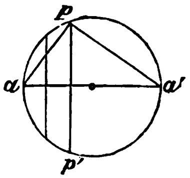
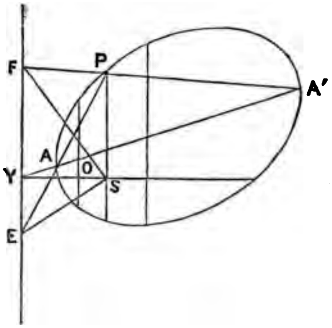


Fig. 83.



Let EF be the given line. Let A′AY be the diameter conjugate to it, meeting it in Y. Then, by Prop. 2, the middle points of a series of chords parallel to EF project into the middle points of another series of chords in the projected conic, also parallel to EF. That is, the diameter AA′ projects into a diameter *aa′*. Take any point S on a line through Y perpendicular to EF as focus of projection; then *aa′* will be parallel to SY or perpendicular to EF. Hence the diameter *aa′* cuts its ordinates at right angles; that is, it is an axis of the conic. Next, if P be any point on the conic, and PA, A′P meet EF in the points E, F, and if S be so taken that ESF is a right angle, the angle APA′ will project into a right angle *apa′*, and the conic will therefore be projected into a circle, while the line EF is also projected to infinity.

184. *Cor.* 1.—If any other point q on the circle be the projection of Q on the conic, we notice that since the angle aqa' is a right angle, therefore if QA and A′Q meet the line EF in G and H, the line GH must also subtend a right angle at the same point S. Hence also the rectangle EY . FY is constant for all positions of the point P and equal to SY^2.

Cor. 2.—If EA′ cut the conic in Q, then QA produced must also pass through F.

185. *Ex.* 6.—*If a chord of a conic pass through a fixed point, the tangents at its extremities will intersect on a fixed line.*

First, let O be the fixed point inside, OQQ′ and OPP′ any two chords through O. Let the tangents at Q, Q′ and P, P′ intersect in F and E respectively. Take EF as directrix of projection, and project the conic into a circle. Then clearly PP′ and QQ′ project into diameters, and therefore O projects into the centre of the circle. And since the tangents at the extremities of any diameter of a circle are parallel, therefore those at the extremities of any chord through O intersect in the directrix EF.

Next draw a line through E meeting the conic in RR′. Then RR′ projects into a chord rr' of the circle parallel to the tangents at p and p'; and since the tangents at r, r' intersect on op, therefore those at R, R′ intersect on OP. Hence the theorem is proved both for an external and internal point.

Def.—The locus of the intersection of tangents whose chords of contact pass through a fixed point is called the 'Polar' of that point, and the fixed point is called the 'Pole' of the line.

186. *Ex.* 7.—*If a chord of a conic subtend a right angle at a fixed point* S *on the conic, it always passes through a fixed point, which lies on the normal at* S.

Let PP′ and QQ′ be any two chords which subtend right angles at S. Let them intersect in O. Take S for focus of projection and the polar of O for directrix.

FIG. 84.

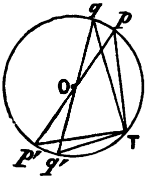
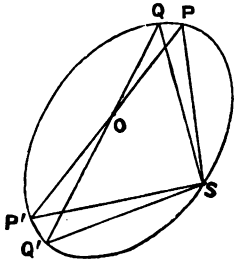


FIG. 85.



Then PP′ and QQ′ project into diameters *pp′* and *qq′*, and, since the angles *p*T*p′* and *q*T*q′* are right angles, the three diameters *op*, *oq*, *o*T are all equal; hence the conic projects into a circle; and since every diameter of the circle subtends a right angle at T, therefore every chord of the conic through O subtends a right angle at S. Again, since *o*T is perpendicular to the tangent at T, therefore OS is perpendicular to the tangent at S; that is, OS is normal.

Cor.—Hence any conic can be projected into a circle by taking a point on its circumference as focus of projection, and the tangent and normal at that point project into tangent and normal.

CHAPTER X.

ORTHOGONAL PROJECTION.

DEFINITION.

187. If perpendiculars be let fall from every point of any figure upon a given plane, the locus of their feet is called the '*Orthogonal Projection*' of that figure upon the given plane.

The given plane is called the *Plane of Projection*.

PROP. 1.

188. *The projection of a straight line is a straight line.*

For if a plane be drawn through the given straight line at right angles to the plane of projection, the line of intersection of these two planes is the projection of the given line.

The *projection of a system of parallel straight lines is a system of parallel straight lines*, for they are the lines of intersection of the plane of projection with a set of parallel planes.

PROP. 2.

189. *If a set of parallel finite straight lines be projected on any plane, the projections are in a constant ratio to the original lines.*

FIG. 86.

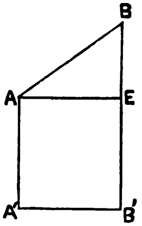
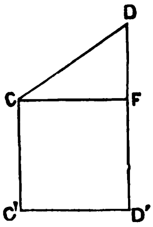


FIG. 87.



Let AB, CD be a pair of parallel straight lines, A′B′, C′D′ their projections. Draw AE, CF parallel to the projections. AA′B′E, CC′D′F are parallelograms, and AE=A′B′ and CF =C′D′.

But AE is parallel to CF and BE to DF.

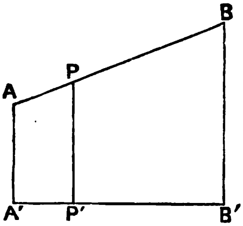
Hence the triangles ABE, CDF are similar

and $$AE : AB :: CF : CD$$
$$\therefore\ A'B' : AB :: C'D' : CD$$

i.e. the ratio of the projected line to the original is constant.

190. *Cor.* 1.—*If a point* P *cut a finite straight line* AB *in a given ratio, the projection* P′ *of* P *cuts the projection* A′B′ *of* AB *in the same ratio.*

FIG. 88.



For from our last proposition we have

$$A'P' : AP :: P'B' : PB$$
and $$\therefore\ A'P' : P'B' :: AP : PB$$ Q.E.D.

191. *Cor.* 2.—*If* P *be the middle point of* AB, *then* P′ *will be the middle point of* A′B′.

192. *Cor.* 3.—It is evident that by properly choosing the plane of projection a line AB can be shortened in any ratio; for if we have only to describe a semicircle on AB and place in it a line AE of the length required, then if we take the plane of projection parallel to AE the projection A′B′ will be equal to AE.

193. *Cor.* 4.—If the plane of projection be taken parallel to any line AB, then the projection A′B′ is equal to AB.

PROP. 3.

194. *The projection of an ellipse is an ellipse.*

Let ABA′ be the given ellipse, *aba′* its orthogonal projection.

FIG. 89.

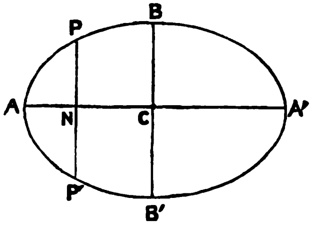
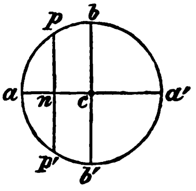


FIG. 90.



The centre *c* projects into the middle point of *aa′* by Prop. 2, Cor. 2.

The middle points N and C of an ordinate PP′ and of the axis BB′ project into the middle points of two parallel lines *pp′* and *bb′* by Props. 1 and 2.

Again

$$\mathrm{PN} : \mathrm{BC} :: pn : bc$$
$$\mathrm{AN} : \mathrm{AC} :: an : ac$$
$$\mathrm{A'N} : \mathrm{AC} :: a'n : ac$$

Hence

$$pn^2 : an \,.\, a'n : bc^2 ; ac^2.$$

Therefore *apa′* is an ellipse. Q.E.D.

195. *Cor.*—An ellipse can be projected into a circle by properly choosing the plane of projection. For describe a semicircle on AC in a plane perpendicular to the plane of the ellipse, and place in it a line CE equal to CB; then if we take a plane parallel to BEB′ as plane of projection, the line AC will project into a line equal to CE or CB, while the length of CB remains unchanged. Conversely, in the same manner as in Prop. 3, the projection of a circle is an ellipse.

PROP. 4.

196. *The projection of a parabola is a parabola.*

For all the ordinates (being parallel lines) project in the same ratio, so also do all the abscissæ. But in the original curve the square of the ordinate varies as the abscissa, whence this relation holds also in the projected curve, which is therefore a parabola.

197. Also, by a proof similar to that given for an ellipse, any hyperbola may be projected into an hyperbola, with its axes in any given ratio, and, as a special case, into a rectangular hyperbola.

198. A diameter of a circle bisects all chords at right angles to it, and projects into a straight line in an ellipse which passes through the middle points of a system of parallel chords. Such a line is called a diameter of the ellipse.

199. Diameters at right angles in a circle project into conjugate diameters in an ellipse, for each is the locus of middle points of chords parallel to the other. Also the converse is true, that if an ellipse be projected into a circle each pair of conjugate diameters project into a pair of diameters at right angles to each other.

200. The projection of the tangent to any curve is a tangent to the projection of the curve; for two ultimately coincident points, and the line joining them, project into two ultimately coincident points and the line joining these projected points.

201. Most propositions in the ellipse which depend on the lengths of a set of parallel lines may be easily proved by Orthogonal Projection. For example, in a circle it is clear that the relation $CV \cdot CT = CP^2$ holds. Therefore it also holds in the ellipse, for all the lines project in the same ratio.

EXAMPLES.

PARABOLA.

1. Prove that the tangent at the extremity of the latus rectum of a parabola passes through the foot of the directrix.

2. Prove that SM bisects the angle PSX.

3. By means of the last example prove that if PSP′ be a focal chord, and PM, P′M′ the perpendicular upon the directrix, the angle MSM′ is a right angle.

4. By means of the last example prove that if PSP′ be a focal chord of a parabola, and PN, P′N′ the ordinates at its extremities,

$$PN \,.\, P'N' = 4AS^2$$

5. By means of the last example prove that if PSP′ be a focal chord of a parabola, and PN, P′N′ the ordinates,

$$AN \,.\, AN' = AS^2$$

6. If PSP′ be any focal chord in a parabola, V its middle point, and VN the perpendicular upon the directrix, then

$$PP' = 2VN$$

7. If PP′ be any chord of a parabola, VN the perpendicular from its middle point upon the axis, then

$$SP + SP' = 2VN$$

8. If PNP′ be a double ordinate of a parabola, and the circle round APP′ cut the axis in the point G, prove that NG is equal to the latus rectum.

9. If the tangents at two points P and P′ of a parabola meet the axis in T and T′, and PN and P′N′ be the ordinates, prove that

$$TT' = NN'$$

10. If the tangents and normals at any two points of a parabola meet the axis in the points T, T′ and G, G′ respectively, prove that

$$GG' = TT'$$

11. If the tangents at the extremities of a focal chord meet the tangent at the vertex in the points Y and Y′, prove that YSY′ is a right angle.

12. If the tangent at a point P meet the tangent at the vertex in the point Y, and if the diameter through Y meet SP in the point Q, prove that

$$SQ = QY$$

13. If QQ′ be any chord of a parabola, and SZ be drawn perpendicular to it meeting the directrix in Z, then the line joining Z to the middle of QQ′ shall be parallel to the axis.

14. If PSP′ be a focal chord, prove that PX and P′X are equally inclined to the axis.

15. If PSP′ be a focal chord of a parabola, and PN, P′N′ be the ordinates to the axis, prove that

$$AN . AN' = AS^2$$

16. If the directrix of a parabola be given, and one point on the curve, prove that the locus of the focus is a circle.

17. If SK be the perpendicular from the focus to the normal at P,

$$SK^2 = AN . SP$$

18. If perpendiculars be let fall from the focus of a parabola upon any three tangents, the feet of these perpendiculars lie in one straight line.

19. If a pair of tangents to a parabola meet the directrix in Z and Z′, prove that the angle ZSZ′ is double of the angle between the tangents.

20. If a pair of tangents to a parabola meet the tangent at the vertex in the points Y and Y′, the angle YSY′ shall be equal to the angle between the tangents.

21. If, in the last example, perpendiculars be drawn from Y and Y′ to the tangents through Y′ and Y respectively, these will intersect on the directrix.

22. QQ′ is a chord of a parabola, RR′ the tangent parallel to it meeting those at Q, Q′ in R and R′. Through R and R′ lines are drawn parallel to the axis meeting the curve in W and W′. Prove that

$$RW=R'W'$$

23. If P be a point on a parabola, and PA meet the directrix in E, and EP′ be drawn parallel to the axis meeting PS produced in P, prove that P′ lies on the parabola.

24. If PSP′ be a focal chord, and PM, P′M′ the perpendiculars upon the directrix, then the lines SM and SM′ are parallel to the tangents at P′ and P respectively.

25. If the tangent at any point P of a parabola meet the directrix in Z, and PM be the perpendicular upon the directrix, prove that the circle round the triangle PMZ passes through the focus.

26. If the normal at a point P meet the axis in G, and PN be the ordinate, show that the triangle PNG is equal to the triangle SPX.

27. If the tangent at P meet the tangent at the vertex in the point Y, then a line drawn through Y parallel to the axis will bisect SP.

28. If the normal at a point P of a parabola meet the axis in G, and a line be drawn through G equal to PG and perpendicular to the axis, its extremity will always lie on a parabola.

29. If the tangents at the extremities of a focal chord cut the latus rectum in D and D′, then

$$SD=SD'$$

30. If the tangents from a point Q to a parabola meet the tangent at the vertex in the points Y and Y′, prove that the angle YSY′ is equal to the angle YQY′.

31. If TP and TP′ be a pair of tangents to a parabola, prove that

$$TP^2 : TP'^2 :: SP : SP'$$

32. The circle on any focal chord as diameter touches the directrix.

33. If two parabolas have the same vertex and axis, and PQN be an ordinate cutting them in P and Q, prove that the ratio of PN to QN is constant.

34. In the last example prove that the tangents at P and Q meet the axis in the same point.

35. If, in the last example, the normals at P and Q meet the axis in G and G′, prove that the length of GG′ is constant.

36. If two equal parabolas have the same axis, prove that any chord of the outer parabola which touches the inner parabola is bisected by it.

37. Given the focus, a point on the curve, and the direction of the axis, draw the tangent and normal to the parabola.

38. Given the vertex, the axis, and a point on the curve, draw the tangent to the parabola.

39. The tangent at P meets the tangent at the vertex in Y. Prove that a circle with centre Y can be described to touch the axis, the focal radius SP, and the perpendicular from P upon the directrix.

40. If TP, TQ be a pair of tangents, prove that a circle with centre T can be described to touch SP, SQ and the perpendiculars from P and Q upon the directrix.

41. If any two tangents meet a third tangent in R and R′, prove that the angle between these tangents is equal to the angle which RR′ subtends at the focus.

42. If PP′ be any chord of a parabola, and OSY be drawn perpendicular to it, meeting the directrix in O, and if PM be the perpendicular upon the directrix, prove that

$$OY^2 - SY^2 = OM^2$$

43. If SY, the perpendicular upon the tangent at a point P of a parabola, cut the curve in Q, and SY′ be the perpendicular upon the tangent at Q, prove that $SY'^2=SM.SY$, where QM is the ordinate of the point Q.

44. Describe a parabola touching a given straight line at a given point and having a given focus.

45. Having given the focus and two tangents to a parabola, construct the curve.

46. Having given four tangents to a parabola, show how to find its focus and directrix.

47. Having given a parabola traced on paper, show how to find its focus and directrix.

48. Describe a parabola touching three given lines, and having its focus in another line.

49. A circle is drawn on the latus rectum of a parabola as diameter, and any chord QPN is drawn parallel to the axis meeting the circle in Q and the latus rectum in N. Prove that

$$QN^2=4AS.PN$$

50. From S, the focus of a parabola, a line is drawn parallel to the tangent at a point P, meeting the curve in Q; the diameter through P meets SQ in E. Show that the locus of E is a parabola.

51. Given in a parabola that $PN^2=4AS.AN$, prove that

$$SP=NX$$

52. If the focal radii of a parabola be bisected, all the points of section lie on a parabola.

53. If the focal radii of a parabola be divided in a constant ratio, the points of section lie on a parabola.

54. If all the lines drawn from the vertex of a parabola to points on the curve be divided in a constant ratio, the points of section lie on another parabola.

55. If two parabolas have a common vertex and axis, and any line be drawn through the vertex cutting them in P and

Q, prove that the ratio of AP to AQ is the same for all such lines.

56. Prove that the tangents at the points P and Q in the last example are parallel.

57. Two parabolas have a common focus and axis; their vertices are A and A′. SPP′ is drawn cutting the curves in P and P′. Prove that AP is parallel to AP′.

58. Prove that in the last example the tangents at P and P′ are parallel.

59. If T be any point on the double ordinate PP′ of a parabola, and the latus rectum cut the auxiliary circle of the point T in E and E′, prove that

$$TP \, . \, TP' = SE \, . \, SE'$$

60. If the tangent and normal at a point P of a parabola meet the directrix in Z and K, prove that

$$PZ^2 : PK^2 :: AS : AN$$

61. The foci and vertices of two parabolas are S, S′ and A, A′. Any two chords AP and A′P′ are drawn making equal angles with the respective axes. Prove that SA′P and S′A′P′ are similar triangles.

62. Two parabolas have a common vertex, and have their axes at right angles. If from any point R in their chord of intersection the ordinates RPN and RQM be drawn to their axes respectively, prove that

$$RP : PN :: RQ : QM$$

63. If a circle be described touching the tangent at the vertex of a parabola and having its centre on the axis, prove that it will not cut the parabola if its diameter is less than the latus rectum.

64. If V be any point on the perpendicular MP from a point P of a parabola upon the directrix, prove that if MS meet the auxiliary circle of the point V in E

$$SP \, . \, PV : SE \, . \, SM :: SP^2 : SM^2$$

or, in other words, is the same for every point V on the line MP.

65. If the normal at a point P meet the axis in G, and QM be the perpendicular from the middle point of PG upon the axis, prove that

$$QM^2 = AS . SM$$

66. The tangent at a point P of a parabola meets the tangent at the vertex in the point Y, and the diameter through Y meets SP in Q. Prove that the locus of Q is a parabola.

67. If PSP′ be a focal chord, and the tangents at P and P′ meet the latus rectum in D and D′, prove that

$$SP . SP' = SD . SD'$$

68. If PSP′ be a focal chord of a parabola, and PN, P′N′ the ordinates, prove that

$$SP' : SP :: AS : AN$$

and
$$SP : SP' :: AS : AN'$$

69. From the last example prove that if SP be double of SP′ the ordinate of P′ will bisect AS.

70. If the tangent at P meet the directrix in Z, and PN be the ordinate, prove that

$$SZ^2 : SP^2 :: AS : AN$$

71. If the tangent and normal at a point P meet the axis in T and G, and GR be drawn perpendicular to the normal to meet PS produced in R, prove that PTRG is a rectangle.

72. If QQ′ be a chord of a parabola, and the tangent parallel to it meet the tangents at Q and Q′ in the points R and R′, and if these tangents intersect in T, prove that the triangle TQQ′=4TRR′.

73. The tangents drawn to a parabola from any point make equal angles respectively with the axis and focal radius drawn to that point.

74. If QVQ′ be a double ordinate to a diameter PV of a parabola, and the tangent at P meet the tangent at the vertex

in the point Y, and if QD be drawn perpendicular to PV, prove that QVD and SAY are similar triangles.

75. From the last example prove that

$$QD^2 : QV^2 :: AS : SP$$

76. From the last example prove that

$$QD^2 = 4AS . PV$$

77. If from any point in the latus rectum of a parabola a pair of tangents be drawn meeting the curve in P and Q, prove that the semi-latus rectum is a harmonic mean between SP and SQ.

78. PQ is a focal chord of a parabola meeting the directrix in Z; QM, PN are the ordinates. If ZM meet PN produced in R, prove that

$$RP = PN$$

79. If AP, AQ be two chords drawn from the vertex of a parabola at right angles to one another, and PN, QM be ordinates, prove that

$$PN . QM = AN . AM = 16AS^2$$

and that the chord PQ passes through a fixed point.

80. If a circle cut a parabola in the points P, Q, P′, Q′, show that the six chords joining these points are two and two equally inclined to the axis.

81. By means of the last example show that if a circle cut a parabola in P′, Q′, and touch the parabola in P, then PQ′ and PP′ are equally inclined to the axis.

82. If from any point P of a parabola perpendiculars PN, PM be drawn on the axis and tangent at the vertex, show that the line MN always touches another parabola.

83. KL is any line perpendicular to the axis of a parabola, and from any point P on it a tangent PQ is drawn to the parabola, and a line PT making the same angle with PQ as PQ makes with KL. Show that PT envelopes a circle.

84. If two tangents to a parabola meet in T, and S be the focus, and from any point in ST perpendiculars be drawn to the

tangents, the line joining the feet of these perpendiculars is parallel to the directrix.

85. If three parabolas be described touching the sides of a triangle, the triangle formed by joining their foci is greatest when it is equilateral.

86. If QQ′ be any chord of a parabola meeting the axis in T, and PN be drawn from its middle point perpendicular to the axis, prove that

$$PN : 2AS :: TN : PN$$

87. By means of the last example prove that the locus of the middle points of a system of parallel straight lines is a straight line parallel to the axis.

88. A chord PP′ of a parabola is normal to the parabola at P, and PSP′ is a right angle. Prove that SP′ is double of 'SP.

89. If two parabolas have the same vertex and axis, and any two lines AQP and AQ′P′ be drawn through the vertex cutting them in the points Q, P and Q′, P′, prove that QQ′ and PP′ subtend equal angles at the foci.

90. If TP and TQ be the tangents at the extremities of a chord TQ which is normal at P, and PN be the ordinate to the axis, prove that

$$PQ : PT :: PN : AN$$

91. Prove that the intersection of tangents to a parabola which cut at a given angle lie on a hyperbola having the same focus and directrix.

92. Prove that chords of a parabola which subtend a given angle at the focus all touch an ellipse having the same focus and directrix.

93. GK is drawn from the foot of the normal PG at a point P of a parabola perpendicular to SP cutting the circle on SP as diameter in L; LS produced meets tangent at P in the point O. Show that the ratio of OS to OP is constant.

94. If QV, Q′V′ be ordinates to a diameter PVV′, and the chord QQ′ meet this diameter in T, and if *qv* be drawn parallel

to the ordinates from the middle point of QQ′ to meet the diameter in v, prove that

$$qv^2=2SP.Tv$$

95. If PSP′ be a focal chord of a parabola and Q its middle point, and QM be the perpendicular to the axis, then SQ shall be equal to MN and

$$QM^2=2AS.AM$$

96. If POQ, P′OQ′ be any two chords of a parabola, pq and $p'q'$ the focal chords parallel to them, prove that

$$OP.OQ : OP'.OQ' :: pq : p'q'$$

97. If PP′ be a chord of a parabola which meets the directrix in Z, and if the tangents at P and P′ intersect one another in T, prove that ZST is a right angle.

98. E and F are two fixed points on a parabola, P any other point on it. If PE and PF meet the directrix in Z and Z′, prove that the angle ZSZ′ is constant.

99. If PSQ be a focal chord of a parabola, AP′ a parallel chord through the vertex A meeting the latus rectum in Q′, show that $AP'.AQ'=PS.SQ$; and if PM, QM be drawn perpendicular to the axis, show that $AP'=MN$ and $SM\sim SN=$ the abscissa of P′.

100. A point V is taken on an ordinate PM produced of a parabola, and ME is taken on MP a mean proportional between MP and MV. If the diameters through E and V meet the curve in R and Q, prove that PQ meets the axis in the foot of the ordinate of R.

101. If a tangent PQ to a parabola cut two tangents OA, OB in P and Q, prove that

$$\frac{OP}{OA}+\frac{OQ}{OB}=1$$

102. ABC is an equilateral triangle. Two parabolas are described, one touching AB, AC in B and C respectively, and the other touching BC, BA in C and A respectively. Show that the parabolas have a common focus, and if they cut in D the tangent at D to either is parallel to the axis of the other.

103. Given four tangents to a parabola, construct it; and prove that the distances of the focus from the four angular points of the quadrilateral formed by the tangents form a proportion.

104. If PQ be a focal chord of a parabola and R any point on the diameter through Q, show that the focal chord parallel to $PR = \frac{PR^2}{PQ}$

EXAMPLES.

ELLIPSE.

1. Prove that the tangent at the extremity of the latus rectum of an ellipse passes through the foot of the directrix.

2. If P be any point on an ellipse whose axis is AA′, and if A′P produced meet the directrix in the point F, prove that SF bisects the angle PSX.

3. If in the last example PA produced meet the directrix in E, prove that FSE is a right angle.

4. If in the last example A′E meet the ellipse in *p*, prove that PS*p* and FA*p* are straight lines.

5. If PS*p* be a focal chord, and PN, *pn* the ordinates to the axis of the ellipse, prove that

$$PN \cdot pn : AN \cdot An :: SX^2 : AX^2$$

and

$$PN \cdot pn : A'N \cdot A'n :: SX^2 : A'X^2$$

6. If PNP′ be a double ordinate of an ellipse, and the circle round the triangle PA′P′ cut the axis in the point G, prove that the ratio of NG to AN is constant.

7. If the tangents at two points P and P′ meet the axis in T and T′, and PN, P′N′ be the ordinates, prove that

$$TT'^2 : NN'^2 :: CT \cdot CT' : CN \cdot CN'$$

8. If the normal at a point P of an ellipse meet the axis in G, and PN be the ordinate, prove that

$$CG : CN :: CS^2 : CA^2$$

9. In the last example prove that

$$CG : CS :: CN : CX$$

10. By means of the last example prove that CP, and SY the perpendicular upon the tangent at P, intersect on the directrix.

11. Prove that the circles round the triangles SPN and XPG intersect on the central radius CP.

12. A triangle ABC is inscribed in an ellipse so that the tangents at the angular points are parallel to the opposite sides. Prove that the normals at A, B, and C meet in a point.

13. In the last example prove that the line joining any angular point to the intersection of tangents at the other two passes through the centre, and the portion on any tangent intercepted by the other two tangents is bisected at the curve.

14. If the tangents at the extremities of a focal chord meet the tangent at the vertex in Y and Y′, prove that YSY′ is a right angle.

15. If P be any point on an ellipse, and PA produced meet the directrix in the point E, prove that the tangent at P meets the tangent at A′ on the line ES produced.

16. By means of the last example prove that the tangents at the extremities of any focal chord form with the tangents at the extremities of the axis a quadrilateral whose diagonals intersect at right angles in the focus.

17. Prove that if the tangents at the extremities of a focal chord intersect the tangent at A′ in the points Z and Z′, then

$$A'Z \cdot A'Z' = A'S^2$$

18. If the tangent at a point P meet the tangent at the vertex A in Y, prove that SY and A′P meet on the directrix.

19. If the tangents at the extremities of a focal chord meet the axis in T and *t*, prove that

$$AT \cdot At : A'T \cdot A't :: AS^2 : A'S^2$$

20. If the tangent at P meet the tangents at the vertices in Y and Z, prove that CY is parallel to A′P and CZ to AP.

21. If A′P meet the tangent at the vertex in R, prove that the tangent at P bisects AR.

22. From the last example show how to draw the tangent at a point P when the axis AA′ is given.

23. If the tangent at P meet the tangent at the vertex in Y, and the diameter through Y meet SP in Q, prove that the ratio of SQ to SP is constant.

24. In the last example prove that the locus of Q is an ellipse.

25. If the tangent at P meet the tangents at A and A′ in Y and Z, and PN be the ordinate, prove that

$$PN : A'N :: AY : AC$$

and

$$PN : AN :: A'Z : AC$$

26. From the last example prove that

$$AY \, . \, A'Z = BC^2$$

27. If P*p* be a focal chord, and the tangents at P and *p* meet the tangent at the vertex A in Y and Y , prove that YY′ subtends the same angle at the centre as P*p* subtends at A′.

28. If the tangent at P meet the tangents at A and A′ in the points Y and Z, prove that the locus of the intersection of CY and SZ is an ellipse whose axis is CS.

29. If the ordinate PM to the conjugate axis meet the circle on the conjugate axis as diameter in Q, prove that

$$QM : PM :: BC : AC$$

30. If the ordinate NP meet the circle on the transverse axis in Q, and PG be drawn parallel to CQ meeting the axis in G, prove that PG=BC.

31. If the ordinate PN at any point P of an ellipse be produced to meet the circle round the triangle APA′ in Q, prove that the locus of Q is an ellipse.

32. Show how to find the centre of a given ellipse.

33. If any parallelogram circumscribe an ellipse the diagonals are conjugate diameters.

34. If any ordinate MP be produced to meet the tangent at the extremity of the latus rectum in Q, prove that QM=SP.

35. Prove that the transverse axis is the greatest chord which can be drawn in an ellipse.

36. If Q be a point outside an ellipse, prove that

$SQ+S'Q>AA'$, and if Q be inside $SQ+S'Q<AA'$

37. Find where the angle SPS′ is a maximum.

38. Given one focus of an ellipse, and a point on the curve, and the length of transverse axis, prove that the locus of the other focus is a circle.

39. Show that the circle described on any focal radius as diameter touches the circle on the transverse axis.

40. If Q, Q′ be any two points on an ellipse, P any other point on it, and if QP, Q′P meet the directrix in q, q', prove that the angle $QSQ'=2qSq'$.

41. Find the locus of the centre of a circle which touches internally one fixed circle and another externally.

42. The radii from the focus to the ends of a diameter make equal angles with the tangents at those points.

43. If CD be the semidiameter conjugate to CP, and DQ be drawn parallel to SP and CQ perpendicular to DQ, then $CQ=BC$.

44. If CY, AZ be the perpendiculars from the centre and extremity of axis on tangent to an ellipse at any point P, and N the foot of the ordinate, then $CA \,.\, AZ=NA \,.\, CY$.

45. TYLZ is a tangent at L, the extremity of the latus rectum meeting the axis in T and the circle on AA′ in Y and Z. Prove that YL : YZ :: semi-latus rectum : trans. axis.

46. The loci of the centres of the escribed circles of the triangle SPS′ opposite to the angles S and S′ are the tangents at the vertices.

47. Prove that the product of the radii of the circles in the last example is equal to BC^2.

48. The locus of the centre of the escribed circle of the triangle SPS′ which is opposite to P is an ellipse having SS′ for its minor axis.

49. If the tangent at a point P meet the axes in T and t such that $CT=Ct$, prove that

$$PT : Pt :: BC^2 : AC^2$$

50. If the tangent at a point P of an ellipse be parallel to AB, and PN, PM be ordinates to the axes, prove that MN is parallel to AB.

51. In an ellipse a circle is described on NT as diameter. Show that it cuts the circle on AA′ as diameter orthogonally.

52. A circle cannot intersect an ellipse in more than four points.

53. A and A′ are the extremities of the transverse axis of an ellipse; T is the point where the tangent at P meets the axis; through T a straight line is drawn at right angles to AA′ meeting AP and A′P produced at Q and R. Prove that $QT=RT$.

54. If the tangents at A and B intersect in O, and PCP′ be the diameter which passes through O, prove that $OP \, . \, OP'=CP^2$.

55. An ellipse cuts the sides of a triangle ABC in the points 1, 2, 3, 4, 5, 6. Prove that

$$B1 \, . \, B2 \, . \, C3 \, . \, C4 \, . \, A5 \, . \, A6=B6 \, . \, B5 \, . \, C2 \, . \, C1 \, . \, A4 \, . \, A3$$

56. The diagonals of any parallelogram which circumscribes an ellipse pass through the centre.

57. ABDE is any quadrilateral circumscribing an ellipse. If CA, CB, CD, CE meet the chords of contact in A′, B′, C′, E′ respectively, A′B′D′E′ will always be a parallelogram.

58. If TQ, TQ′ be a pair of tangents drawn from T to an ellipse, prove that TQQ′ and CPP′ are similar triangles, CP and CP′ being the semidiameters parallel to the tangents.

59. If the tangents at Q and Q′ meet the tangent parallel to QQ′ in R and R′, and CR, CR′ meet the ellipse in L and L′, prove that LL′ is parallel to QQ′.

60. If the ordinate PN meet the diameter conjugate to CP in n, prove that $PN \, . \, Pn=BC^2$.

61. In the last example if nk be drawn perpendicular to CP and meeting the axis in G, then PG is the normal at P.

62. CP and CD are conjugate semidiameters; PQ is a chord parallel to one of the axes. Show that DQ is parallel to one of the straight lines which join the ends of the axes.

63. If the ordinate NP of an ellipse produced meet the circle on AA′ as diameter in Q, prove that

$$SQ : QX :: SA : AX$$

64. Having given in an ellipse that

$$PN^2 : AN . A'N :: BC^2 : AC^2$$

deduce by means of the last example that

$$SP : PM :: SA : AX$$

65. If SY, S′Y′ be the perpendiculars from the foci on the tangent at P, and PN be the ordinate, prove that

$$PY : PY' :: NY : NY'$$

66. An isosceles triangle is described having each angle at the base double of the third angle; with vertex and one extremity of the base as foci an ellipse is described passing through the other extremity. Prove that

$$AX = SL \text{ (the semi-latus rectum)}$$

67. If CR, SY, S′Y′ be perpendiculars upon the tangent to an ellipse at a point P such that CR=CS, show that R lies on the tangent at B, that the perpendicular from R on SS′ will divide it into two parts equal to SY and S′Y′, and that

$$SP : S'P :: SR^2 : S'R^2$$

68. If QQ′ be a chord of an ellipse, and SZ be drawn perpendicular to it to meet the directrix in Z, then CZ will bisect QQ′.

69. If PSP′ be a focal chord, prove that

$$SP : SP' :: PX : P'X$$

70. If the directrix of an ellipse, one point on the curve, and its eccentricity be given, the locus of one focus is a circle.

71. If SK and S′K′ be the perpendiculars from the foci upon the normal at the point P, prove that

$$SK . S'K' = CA^2 - CP^2$$

and
$$PK . PK' = BC^2$$

72. The perpendicular from S upon the tangent at a point P of an ellipse meets S′P in the point Q. Prove that the locus of Q is a circle.

73. If OP, OP′ be a pair of tangents to an ellipse, and a confocal ellipse be described passing through the point O, prove that the tangent to it bisects the angle between OP and OP′.

74. A pair of tangents to an ellipse cut off a constant arc *y*Y from the circle on the transverse axis. Prove that the orthocentre of the triangle which the chord *y*Y forms with the tangents describes a circle.

75. If P be a point on an ellipse, and PA meet the directrix in E, and if EA′ be joined meeting PS produced in *p*, prove that *p* lies on the ellipse.

76. If TP, TP′ be a pair of tangents to an ellipse, and Q*q*, Q′*q*′ the focal chords parallel to them, prove that

$$TP : TQ :: Qq : Q'q'$$

77. If any chord AQ meet the conjugate axis in R, and CP be the semidiameter parallel to it, prove that

$$AQ . AR = 2CP^2$$

78. If CQ be conjugate to the normal at P on an ellipse, then CP shall be conjugate to the normal at Q.

79. The normal to an ellipse at a point P meets the axis in G, and through G a line is drawn parallel to the tangent meeting SP in K. Prove that PK is a harmonic mean to SP and S′P.

80. If tangents be drawn from any point on an ellipse to touch a confocal ellipse, they are equally inclined to the tangent at that point.

81. If perpendiculars be drawn from the centre of an ellipse on two tangents at right angles to each other, prove that

the semidiameters equal in length to these perpendiculars are conjugate, and that the ordinates to the transverse axis through their extremities meet the perpendiculars on the circle on AA′.

82. If the normal at the extremity of the latus rectum pass through the extremity of the conjugate axis, find the eccentricity.

83. If SY, S′Y′ be the perpendiculars on the tangent at P, then SY′ and S′Y will intersect on the normal at P.

84. If SY, S′Y′ be the perpendiculars on the tangent at P, and if the normal at P meet the axis in G, then SY′ or S′Y will bisect PG.

85. From any point P on an ellipse PM, PN are drawn perpendicular to the equiconjugate diameters. Show that the normal at P bisects MN.

86. The straight line parallel to the transverse axis which passes through the intersection of normals at the end of a focal chord bisects that chord.

87. If CP, CQ be semidiameters of an ellipse at right angles to one another,

$$\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{AC^2}+\frac{1}{BC^2}$$

88. If the tangent and normal at a point P of an ellipse meet the axis in T and G, prove that

$$CG \, . \, CT = CS^2$$

89. PCP′ is any diameter of an ellipse. The tangents at any two points D and E intersect in F; PE and P′D intersect in G. Show that FG is parallel to the diameter conjugate to PCP′.

90. Any focal chord is divided harmonically at the focus and directrix.

91. CP and CD are any conjugate semidiameters of an ellipse; PQ is a chord parallel to a fixed straight line. Show that DQ will also be parallel to a fixed straight line.

92. A pair of conjugate diameters being given, show how to find the axes.

93. From the ends P and D of conjugate diameters in an ellipse draw straight lines parallel to any tangent; and from centre C draw any straight line cutting these straight lines and the tangent in p, d, t respectively. Then show that

$$\mathrm{C}p^2+\mathrm{C}d^2=\mathrm{C}t^2$$

94. A chord PP′ of an ellipse subtends a constant angle at the focus which is less than BSB′. Prove that the tangents at P and P′ intersect on an ellipse.

95. The normal at P meets the axis in G. Prove that the locus of the middle point of PG is an ellipse.

96. From any point in one of the equiconjugate diameters a pair of tangents TP, TQ are drawn to the ellipse. Prove that a circle will pass round the quadrilateral TPCQ.

97. If the tangent at the extremity of the conjugate axis meet the latus rectum in Q, and if QP be the other tangent drawn from Q to the ellipse, prove that

$$\mathrm{SP}^2=\mathrm{SN}\,.\,\mathrm{NX}$$

where PN is the ordinate of P.

98. If the tangent at a point P be equally inclined to the axes, and PN be its ordinate,

$$\mathrm{PN}\,.\,\mathrm{CS}=\mathrm{BC}^2$$

99. P is any point on an ellipse, AA′ any diameter. If A′P and AP meet the diameter CD conjugate to CA in the points M and N, then

$$\mathrm{CM}\,.\,\mathrm{CN}=\mathrm{CD}^2$$

100. Two ellipses have a common focus S; QSq is a chord meeting them respectively in Q and q; the tangents at Q and q intersect in O. If a second pair of tangents OP and Op meet these ellipses respectively in p and P, prove that the angles QSP and qSp are equal.

101. A circle whose radius is equal to the semi-latus rectum touches an ellipse at the point A, and cuts the axis in G; any

chord ApP cuts the circle and ellipse in p and P respectively. Prove that Gp and A′P intersect on the tangent at A.

102. The ordinate of P and the radius CP meet the circle on AA′ in the points p and q; the perpendicular from q upon the axis cuts the ellipse in Q. Prove that the tangent at Q is perpendicular to Cp and cuts off from it a part CR equal to CP.

103. If Q be a point on the major axis AA′ of an ellipse of which S and S′ are the foci, O the centre, and BB′ the minor axis, and if P be a point on the ellipse such that OP is equal to BQ, prove that AQ=SP and A′Q=S′P.

104. If from the extremities of the axes of an ellipse any four parallel straight lines be drawn, the points at which they cut the curve will be the extremities of conjugate diameters.

105. If a point move in such a manner that the sum of its distances from two fixed points is constant, prove that its distance from any one bears a constant ratio to its distance from some fixed line.

106. Part of an ellipse is cut off by a given line LM. Find a point P on the ellipse so that the angle LPM may be a maximum or a minimum.

107. A series of parallelograms is inscribed in an ellipse whose sides are parallel to the equiconjugates. Prove that the sum of the squares of their sides is constant.

EXAMPLES.

HYPERBOLA.

1. LKX is a right-angled triangle; a string whose length is equal to its base has one end fastened to the extremity L of the base, and the other end to a fixed point S; the side KX is made to slide along a fixed straight line, while the string is pressed against the base by a point of a pencil at P. Show that P describes an hyperbola.

2. If a series of arcs of circles be described on a line, their points of trisection lie on an hyperbola.

3. The locus of the centre of a circle which touches two fixed circles externally is an hyperbola.

4. Prove that the asymptote to an hyperbola continually gets nearer to the curve.

5. Given the asymptotes and a focus, find the directrix.

6. Given the asymptotes and the transverse axis, find the focus.

7. If an ellipse and hyperbola have the same foci they intersect at right angles.

8. If an ellipse and hyperbola have the same axes, the asymptotes coincide with the equiconjugate diameters of the ellipse.

9. If the tangent at P meet the asymptote in t, and SP produced meet the same asymptote in Q, prove that

$$SQ=Qt$$

10. If from any point P of an hyperbola a line PK be

drawn parallel to the asymptote meeting the directrix in K, then

$$SP = PK$$

11. The straight lines joining the points where two tangents meet the asymptotes are parallel.

12. The asymptote meets the directrix in D, and the tangent at the vertex in L. Prove that AD is parallel to SL.

13. In the hyperbola prove that

$$SX \,.\, CS = SA \,.\, SA'$$

14. Given the transverse axis and the focus, find the asymptotes of the hyperbola.

15. From the ends P and D of conjugate diameter lines are drawn parallel to the asymptote which lies between them cutting the axis in p and d. Prove that

$$Cp = Cd$$

16. The centre of the inscribed circle of the triangle SPS′ lies on the tangent at the vertex.

17. In the hyperbola a pair of conjugate diameters meet the directrix in Z and Z′. Prove that the rectangle ZX . Z′X is constant.

18. Any tangent meets a pair of parallel tangents on conjugate diameters.

19. PSp is a focal chord, Q any point on the hyperbola; QP, Qp meet the directrix in Z and Z′. Prove that ZZ′ subtends a right angle at S.

20. Prove that the tangents from the centre of an hyperbola meet the curve at an infinite distance.

21. SY is the perpendicular upon the tangent at a point P, and CD is the semidiameter conjugate to CP. Prove that

$$SY : SP :: BC : CD$$

22. If hyperbolas whose transverse axes are equal have a common centre and asymptote, their foci all lie in the same

straight line, their directrices all pass through a fixed point, and their vertices all lie on the same circle.

23. The radius of the circle which touches an hyperbola and its asymptotes is equal to that part of the latus rectum which is intercepted between the curve and its asymptote.

24. PCP′ is any diameter of an hyperbola; an ellipse whose centre is C touches the hyperbola at P and P′. From any point V in CP a pair of tangents are drawn to the ellipse; their chord of contact cuts CP in V′. Prove that if a pair of tangents be drawn to the hyperbola from V′ their chord of contact passes through V.

25. Tangents to an hyperbola are drawn from any point on its conjugate. Show that the chord of contact will also touch the conjugate.

26. Given two points on a parabola and the direction of its axis, prove that the locus of its focus is an hyperbola.

27. PG is the normal at P; S is the focus. Prove that if SG=PG then SP is equal to the latus rectum.

28. A tangent at any point P of an hyperbola meets the asymptotes in T, T′, and conjugate hyperbola in Q, Q′. Show that

$$QQ' : TT' :: \sqrt{2} : 1$$

29. CP and CD are two semiconjugate diameters of an hyperbola, and the normals at P and D intersect in B. Then CB is perpendicular to an asymptote.

30. If a point move in such a manner that the difference of its distances from two fixed points is constant, prove that its distance from any one of them bears a constant ratio to its distance from some fixed line.

31. If SY be the perpendicular on the tangent at P, and CY meet the normal at P in R, PR is equal to SY.

32. A circle is described on the latus rectum of an hyperbola as diameter, and the common tangent to the conic and circle touches the conic in P and the circle in Q. Prove that the angle PSQ is bisected by the latus rectum.

33. If any tangent to an hyperbola meet the asymptotes in R and R′, prove that the angle RSR′ is the supplement of half the angle between the asymptotes.

34. A rectangle circumscribes an hyperbola. Prove that the difference of the sides of the parallelogram formed by joining the points of contact is equal to the diagonal of the rectangle.

35. If in an hyperbola a circle be described passing through C and X, the tangent to this circle from the focus is equal to the transverse axis.

36. Given a focus, a tangent, and the eccentricity of an hyperbola, find the locus of its centre.

37. A point E moves on an arc of a circle; S is a fixed point outside the circle. Prove that the envelope of a straight line through E at right angles to SE is a conic.

38. Tangents are drawn from a point M to an hyperbola, and lengths MO, MO′ are taken on them equal to the focal distances of M. Show that OO′=the transverse axis.

39. Given a straight line and two points on the same side of it, find geometrically the transverse axis of the hyperbola which shall have the points as foci and the straight line for normal.

40. From the foci of an hyperbola perpendiculars are let fall to the tangent at any given point of the curve. With the feet of these perpendiculars as foci an hyperbola is described touching the transverse axis of the given hyperbola. Prove that the point in which it touches the axis will be the foot of the ordinate of the given point, and that the hyperbola described will be similar to the given one.

41. If P be any point of an hyperbola whose foci are S and H, and if the tangent at P meet an asymptote in T, the angle between that asymptote and HP is double the angle STP.

42. If PP′ be a chord of an hyperbola parallel to the transverse axis, and the two circles be drawn through a focus S touching the curve at P and P′ respectively, prove that F, the second point of intersection of the circles, will be at the in-

tersection of PP′ and ST, where T is the point of intersection of the tangents at P and P′.

43. If from three points on one branch of an hyperbola, viz. the vertex and any two points equidistant from it, three parallel straight lines be drawn to its asymptotes, they will be in continued proportion.

THE RECTANGULAR HYPERBOLA.

1. From points in the diameter of a circle produced perpendiculars are erected equal to the lengths of the tangents from those points. Prove that their extremities lie on a rectangular hyperbola.

2. Find the locus of the middle point of a line cutting off a constant area from the corner of a square.

3. If the ordinate PN of an hyperbola be produced to meet the rectangular hyperbola which has the same transverse axis in Q, prove that the tangents at P and Q meet on the axis.

4. What is the eccentricity of a rectangular hyperbola?

5. If P be any point on a rectangular hyperbola, CP is a geometric mean between SP and S′P.

6. In the rectangular hyperbola the tangents and normals at the extremities of two conjugate diameters intersect on an asymptote.

7. PQ, P′Q′ are any two parallel chords of a rectangular hyperbola, and AB any chord perpendicular to them. Show that the angle subtended by QQ′ at A is equal or supplementary to that subtended by PP′ at B.

8. The locus of the centres of all equilateral hyperbolas described about an equilateral triangle is the inscribed circle.

9. In the rectangular hyperbola if QVQ′ be the double ordinate of the diameter P′CP, and the circle round the triangle P′QQ′ cut the diameter in *p*, prove that QPQ′P′ is a parallelogram.

10. In a rectangular hyperbola N is the foot of the ordinate and G the foot of the normal. Show that N bisects CG.

11. With the focus S of a rectangular hyperbola as centre, and radius equal to CS, a circle is described. Prove that it will touch the conjugate hyperbola.

12. AB, CD are two chords of a rectangular hyperbola on opposite branches, I and K are their middle points, and O the centre; through I and K are drawn two other chords parallel to CD and AB intersecting in L. Show that OILK is inscribable in a circle.

13. If two chords of a rectangular hyperbola be at right angles, each of their four extremities is the orthocentre of the triangle formed by the other three.

14. If four such points be given, and the tangent at one of them, find the centre.

EXAMPLES.

SECTIONS OF THE CONE.

1. The eccentricities of all sections made by parallel planes are the same.

2. Show how to cut from a cone an ellipse of given eccentricity.

3. Show how to cut from a cone an ellipse whose axis shall be of given length.

4. Find under what conditions it is possible to cut a rectangular hyperbola from a given cone.

5. Show how to cut from a given cone a parabola of given latus rectum.

6. Show that the foci of all parallel parabolic sections lie on a straight line.

7. A parabola being given, prove that the locus of the vertices of right circular cones of which it could be a section is a parabola whose focus is at the vertex of the given parabola.

8. In what position must a candle be held in order that the shadow of a sphere on a plane may be a parabola?

9. A parabola being given, prove that the axes of all right circular cones of which it could be a section envelope a parabola whose focus is the vertex of the given parabola, and whose latus rectum is equal to that of the given parabola.

10. If two cones, whose vertices are V and V′, intersect in a parabola whose vertex is A, prove that the difference of the angles of the cones is equal to the angle VAV′.

11. Show that the foci of all parallel elliptic sections of a cone lie on a straight line.

12. An ellipse being given, prove that the vertices of all right circular cones of which it could be a section lie on an hyperbola whose foci are at the vertices of the ellipse and vertices at the foci.

13. How must a candle be held so that the shadow of a sphere on a plane may be elliptic?

14. An ellipse being given, prove that the axes of all right circular cones of which it could be a section envelope an hyperbola.

15. If VN be the perpendicular from the vertex of a cone on the elliptic section AA′ whose foci are S, and if the axis of the cone meet AA′ in T,

$$CT . CN = CS^2$$

16. In the last example prove that

$$VN^2 : CN^2 - CS^2 :: CA^2 - CS^2 : CA^2$$

17. By means of Example 14 show how to find the axis of a right circular cylinder of which it is a section.

18. Show that the product of the perpendiculars from the vertices of an elliptic section upon the axis of the cone is equal to the square of the minor axis.

19. Prove that the minor axis of the hyperbola in Example 14 is equal to that of the ellipse.

20. Prove by means of the cone that in an ellipse

$$SP + S'P = AA'$$

21. If V be the vertex of a cone, and PCP′ any diameter of an elliptic section,

$$VP + VP' = \text{constant}$$

22. If the vertical angle of the cone be a right angle, and the section be an ellipse, prove that the major axis is equal to the difference of the radii of the spheres, and that the minor axis is a mean proportional between their diameters.

23. A right cone is cut by a plane, and the section projected on a plane passing through the vertex, and perpendicular to the axis of the cone. Prove that the projection is a conic whose focus and directrix are the vertex and the intersection of the plane section with the plane of projection.

24. In a given plane are drawn a series of confocal conics, upon which stand cones with their vertical angles right angles. Show that the locus of their vertices is given by the intersection of an hyperbola whose vertices are the foci of the conics and a circle concentric with the hyperbola and passing through its foci.

EXAMPLES

CURVATURE.

1. Prove that the radius of curvature at the vertex of a parabola is equal to the semi-latus rectum.

2. In a parabola L is the extremity of the latus rectum; the normal at L meets the curve again in Q. Prove that LQ is the diameter of the circle of curvature at L.

3. The radius of curvature at the vertex of an ellipse or hyperbola is equal to the semi-latus rectum.

4. The common chord of the circle of curvature and a conic at any point makes the same angle with the axis as the tangent at that point.

5. If a point P be taken on an ellipse such that the common chord of the circle of curvature and the ellipse is the normal at P, and if PM be the ordinate to the conjugate axis, prove that

$$AB \,.\, CM = BC^2$$

6. Prove the following construction. Erect a perpendicular to the normal at the point where it meets the axis, and again at the point Q where this perpendicular meets the focal radius; draw CQ perpendicular to it meeting the normal at C: then C is the centre of curvature.

7. Prove that the common chord of a parabola and its circle of curvature at any point touches another parabola.

8. If MM′ be a focal chord of a parabola, and R, R′ the radii of curvature at M and M′, and $2p$=the latus rectum, prove that

$$\frac{1}{R} + \frac{1}{R'^{\frac{2}{3}}} = \frac{1}{p^{\frac{2}{3}}}.$$

9. If the circle of curvature at a point of an ellipse pass through one of the foci, show that the distances of that point from the foci are as 1 to 3.

10. The chord of curvature at any point through the focus is equal to the focal chord parallel to the tangent at the point.

11. At a point on a rectangular hyperbola the diameter of the circle of curvature is the common chord of the hyperbola and circle.

12. The radius of curvature at a point P of a rectangular hyperbola varies as CP^3.

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